GAIN OF ANALYTICITY FOR SEMILINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We discuss gain of analyticity phenomenon of solutions to the initial value problem for semilinear Schrödinger equations with gauge invariant nonlinearity. We prove that if the initial data decays exponentially, then the solution becomes real-analytic in the space variable and a Gevrey function of order 2 in the time variable except in the initial plane. Our proof is based on the energy estimates developed in our previous work and on fine summation formulae concerned with a matrix norm.

1. INTRODUCTION

In this paper we study the gain of regularity phenomenon of solutions to the initial value problem for semilinear Schrödinger equations of the form:

$$\partial_t u - i\Delta u = f(u, \partial u) \quad \text{in} \quad (-T,T)\times\mathbb{R}^n, \tag{1}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \mathbb{R}^n, \tag{2}$$

where $u(t,x)$ is a complex-valued unknown function of $(t,x) = (t,x_1,\dots,x_n) \in [-T,T]\times\mathbb{R}^n$, $T>0$, $i=\sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j=1,\dots,n)$, $\partial = (\partial_1,\dots,\partial_n)$, $\Delta = \partial_1^2+\cdots+\partial_n^2$ and n is the space dimension. Throughout this paper, we assume that the nonlinearity $f(u,v)$ is a real-analytic function on $\mathbb{R}^{2+2n}$ having a holomorphic extension on $\mathbb{C}^{2+2n}$, and that $f(u,v)$ satisfies

$$f(u,v) = O(|u|^3+|v|^3) \quad \text{near} \quad (u,v)=0,$$

$$f(e^{i\theta}u, e^{i\theta}v) = e^{i\theta}f(u,v) \quad \text{for} \quad (u,v)\in\mathbb{C}^{1+n}, \theta\in\mathbb{R}. \tag{3}$$

For $z=(u,v)\in\mathbb{C}^{1+n}$ and any multi-index $\alpha = (\alpha_0,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^{1+n}$, we denote

$$|\alpha| = \alpha_0+\cdots+\alpha_n, \quad z^\alpha = u^{\alpha_0}v_1^{\alpha_1}\cdots v_n^{\alpha_n}.$$

It follows from our hypothesis on the nonlinearity that $f(z)$ is given by

$$f(z) = \sum_{p=1}^{\infty} \sum_{\substack{|\alpha|=p+1\\|\beta|=p}} f_{\alpha\beta} z^\alpha \bar{z}^\beta, \quad f_{\alpha\beta}\in\mathbb{C}, \tag{4}$$

and that for any $R>0$ there exists $C_R>0$ such that

$$A_p \equiv \sum_{\substack{|\alpha|=p+1\\|\beta|=p}} |f_{\alpha\beta}| \leqslant C_R R^{-(2p+1)}, \quad p=1,2,3\dots.$$

Here we introduce notation. Let θ and l be real numbers. $H^{\theta,l}$ is the set of all tempered distributions on $\mathbb{R}^n$ satisfying

$$\|u\|_{\theta,l}^2 = \int_{\mathbb{R}^n} |\langle x\rangle^l \langle D\rangle^\theta u(x)|^2 dx < +\infty,$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $|x| = \sqrt{x_1^2+\cdots+x_n^2}$ and $\langle D\rangle = (1-\Delta)^{1/2}$. In particular, set $H^\theta = H^{\theta,0}$, $\|\cdot\|_\theta = \|\cdot\|_{\theta,0}$, $L^2 = H^0$ for short. $\|\cdot\|$ and $(\cdot,\cdot)$ denote the L^2-norm and the L^2-inner product

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respectively. In this paper we treat not only scalar-valued functions but also vector-valued ones. The $(L^2)^m$-norm and the $(L^2)^m$-inner product is denoted by the same notation:

$$(u,v)=\int_{\mathbb{R}^n}\sum_{j=1}^m u_j(x)\bar{v}_j(x)dx, \quad \|u\|=\sqrt{(u,u)}$$

for $u={}^t[u_1,\dots,u_m]$ and $v={}^t[v_1,\dots,v_m]$. Let X be a Banach space, and let k be a nonnegative integer. $C^k(I;X)$ denotes the set of all X-valued C^k-functions on the interval I. In particular, set $C(I;X)=C^0(I;X)$ for short. For any real number s, $[s]$ is the largest integer not greater than s.

In the previous paper [1] the author studied the finite gain of regularity of solutions to (1)-(2). Loosely speaking, if $u_0(x)=o(|x|^{-l})$ as $|x|\to\infty$ with some positive integer l, then the solution u gains spatial smoothness of order l locally in x when $t\neq 0$. More precisely, we proved the following.

Theorem 1 ([1]). *Let $\theta>n/2+3$, and let l be a nonnegative integer. Then for any $u_0\in H^{\theta,l}$, there exists $T>0$ depending only on $\|u_0\|_\theta$ such that* (1)-(2) *has a unique solution $u\in C([-T,T];H^\theta)$ satisfying*

$$\langle x\rangle^{-|\alpha|}\partial^\alpha u\in C([-T,T]\setminus\{0\};H^\theta) \quad \textit{for} \quad |\alpha|\leqslant l.$$

This type of properties of dispersive equations have been investigated in the last two decades. See, e.g., the references in [1]. For local existence theorems for more general semilinear Schrödinger-type equations, see [12], [19], [24]. More recently, in [6] Hayashi, Naumikin and Pipolo studied the infinite version of Theorem 1 for one-dimensional equations with small initial data. Roughly speaking, they proved that if u_0 is small and $u_0(x)=o(e^{-|x|})$ as $|x|\to\infty$, then the solution u becomes real-analytic in x for $t\neq 0$. The purpose of this paper is to prove the infinite version of Theorem 1 without smallness condition on the initial data and the restriction on the space dimension. Our main results are the following.

Theorem 2. *Let θ and s be positive numbers satisfying $\theta>n/2+3$ and $s\geqslant 1$ respectively, and let ε be an arbitrary positive number. For any u_0 satisfying $\exp(\varepsilon\langle x\rangle^{1/s})u_0\in H^\theta$, there exist a positive time T depending only on $\|u_0\|_\theta$, and a unique solution $u\in C([-T,T];H^\theta)$ to* (1)-(2). *Moreover there exist positive constants M and ρ such that*

$$\|\langle x\rangle^{-2m-|\alpha|}\partial_t^m\partial^\alpha u(t)\|_\theta\leqslant M(\rho t)^{-(2m+|\alpha|)}m!^{2s}\alpha!^s \tag{5}$$

for $t\in[-T,T]\setminus\{0\}$, $m\in\mathbb{N}\cup\{0\}$, $\alpha\in(\mathbb{N}\cup\{0\})^n$.

Our condition on the Gevrey exponent does not seem to be optimal. Indeed, in [7] Hayashi and Kato studied the case $s=1/2$ for the equation of the form

$$\partial_t u-i\Delta u=f(u),$$

and proved that the solution becomes real-analytic in $([-T,T]\setminus\{0\})\times\mathbb{R}^n$. Moreover, it is interesting that if $\exp(\varepsilon\langle x\rangle^{1/s})u_0\in H^\theta$ for $s\geqslant 1/2$, then $e^{it\Delta}u_0$ satisfies (5). Using this fact, we can construct nonlinear equations whose solutions have the same regularity property. See Section 8 for the detail. For more information about gain of regularity phenomenon of dispersive equations, see [4], [5], [9], [10], [14], [15], [17], [18], [20], [21] and references therein.

Our method of proof of Theorem 2 is basically due to the energy method developed in [1]. We shall show the uniform bound of $\{w_l\}_{l=0,1,2,\dots}$, where

$$w_l={}^t[r^{|\alpha|}\alpha!^{-s}\langle D\rangle^\theta J^\alpha u, \overline{r^{|\alpha|}\alpha!^{-s}\langle D\rangle^\theta J^\alpha u}]_{|\alpha|\leqslant l},$$
$$J_k=x_k+2it\partial_k, \quad J=(J_1,\dots,J_n),$$

and r is a positive constant. (5) immediately follows from the uniform bound of $\{w_l\}$ and the equation (1).

This paper is organized as follows. In Section 2 we present the elementary facts on pseudodifferential operators associated with nonlinearities. Section 3 is devoted to studying fine summation properties used in the uniform estimates later. In Section 4 we refine the energy method for some linear systems in [1].

Section 5 is devoted to the estimates of nonlinearity in Gevrey classes. In Section 6 we we obtain the uniform energy estimates. In Section 7 we complete the proof of Theorem 2. Finally, in Section 8 we give an interesting example of semilinear Schrödinger equations related with the exponent $s = 1/2$.

2. PSEUDODIFFERENTIAL OPERATORS ASSOCIATED WITH NONLINEAR PDES

In this section we recall the Kato-Ponce commutator estimates established in [11], and pseudodifferential calculus developed in [1]. In addition we present some rough estimates associated with the Leibniz formula for pseudodifferential operators with constant coefficients. One can refer to [2] and [23] for the infromation related to this section.

Let m be a real number. S^m denotes the set of all smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leqslant C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}$$

for any multi-indices α and β. For a symbol $p(x,\xi)$, a pseudodifferential operator $p(x,D)$ is defined by

$$p(x,D)u(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} p(x,\xi)u(y)dyd\xi,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. See [8], [13] and [22] for the detail. We first recall pseudodifferential operators with nonsmooth coefficients and their properties needed later. Let $\sigma \geqslant 0$. $\mathscr{B}^\sigma$ is the set of all $C^{[\sigma]}$-functions on $\mathbb{R}^n$ satisfying

$$\|f\|_{\mathscr{B}^\sigma} = \begin{cases} \sup\limits_{x\in\mathbb{R}^n} \sum\limits_{|\alpha|\leqslant\sigma} |\partial^\alpha f(x)| < +\infty & \text{if} \quad \sigma = 0,1,2,\dots, \\ \sup\limits_{x\in X} \sum\limits_{|\alpha|\leqslant[\sigma]} |\partial^\alpha f(x)| & \\ \quad + \sup\limits_{\substack{x,y\in X \\ x\neq y}} \sum\limits_{|\alpha|=[\sigma]} \dfrac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^{\sigma-[\sigma]}} < +\infty & \text{otherwise.} \end{cases}$$

Similarly, $\mathscr{B}^\sigma S^m$ denotes the set of all functions on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\|p\|_{\mathscr{B}^\sigma S^m, l} = \sup_{\substack{\xi\in\mathbb{R}^n \\ |\alpha|\leqslant l}} \|\langle\xi\rangle^{|\alpha|-m}\partial_\xi^\alpha p(\cdot,\xi)\|_{\mathscr{B}^\sigma} < +\infty$$

for $l = 0,1,2,\dots$. $\mathscr{S}$ denotes the set of Schwartz functions on $\mathbb{R}^n$, and L^p denotes the usual Lebesgue space for all $p \in [1,\infty]$. In [16] Nagase introduced larger classes of symbols, and proved the L^p boundedness theorem by his symbol smoothing technique. We make full use of L^2-version of them.

Theorem 3 (Nagase [16, Theorem A]). *Let $q(x,\xi)$ be a function on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that there exist τ and λ satisfying $0 \leqslant \tau < \lambda \leqslant 1$ such that*

$$|\partial_\xi^\alpha q(x,\xi)| \leqslant C_\alpha \langle\xi\rangle^{-|\alpha|},$$
$$|\partial_\xi^\alpha q(x,\xi) - \partial_\xi^\alpha q(y,\xi)| \leqslant C_\alpha \langle\xi\rangle^{-|\alpha|+\tau}|x-y|^\lambda$$

for $|\alpha| \leqslant n+1$. Then

$$\|q(x,D)u\|_{L^2(\mathbb{R}^n)} \leqslant A(q)\|u\|_{L^2(\mathbb{R}^n)}$$

for any $u\in L^2(\mathbb{R}^n)$, where $A(q)$ depends only on

$$\sum_{|\alpha|\leqslant n+1} \sup_{x,\xi\in\mathbb{R}^n} (\langle\xi\rangle^{|\alpha|}|\partial_\xi^\alpha q(x,\xi)|)$$
$$+ \sum_{|\alpha|\leqslant n+1} \sup_{\substack{x,y,\xi\in\mathbb{R}^n \\ x\neq y}} \left(\langle\xi\rangle^{|\alpha|-\tau} \frac{|\partial_\xi^\alpha q(x,\xi) - \partial_\xi^\alpha q(y,\xi)|}{|x-y|^\lambda} \right).$$

Combining Nagase's idea and results, and well-known facts on smooth symbols, one can obtain the fundamental theorem for algebra and the sharp Gårding inequality.

Lemma 4 (Chihara [1, Lemma 2])**.** *Let* $\sigma > 1$*. If* $p_j(x,\xi) \in \mathscr{B}^\sigma S^j$ *for* $j = 0, 1$*, then*

$$p_0(x,D)p_1(x,D) \equiv p_1(x,D)p_0(x,D) \equiv q(x,D), \tag{6}$$

$$p_1(x,D)^* \equiv r(x,D) \tag{7}$$

modulo L^2*-bounded operators, where* $q(x,\xi) = p_0(x,\xi)p_1(x,\xi)$ *and* $r(x,\xi) = \bar{p}_1(x,\xi)$*. More precisely, there exist a positive integer* ν *and* $C > 0$ *such that for any* $u \in L^2$

$$\begin{aligned} \|(p_0(x,D)p_1(x,D) - q(x,D))u\| &\leqslant C\|p_0\|_{\mathscr{B}^\sigma S^0,\nu}\|p_1\|_{\mathscr{B}^\sigma S^1,\nu}\|u\|, \\ \|(p_1(x,D)p_0(x,D) - q(x,D))u\| &\leqslant C\|p_0\|_{\mathscr{B}^\sigma S^0,\nu}\|p_1\|_{\mathscr{B}^\sigma S^1,\nu}\|u\|, \\ \|(p_1(x,D)^* - r(x,D))u\| &\leqslant C\|p_1\|_{\mathscr{B}^\sigma S^1,\nu}\|u\|. \end{aligned}$$

Lemma 5 (Chihara [1, Lemma 3])**.** *Suppose that* $p(x,\xi) = [p_{ij}(x,\xi)]_{i,j=1,\dots,l}$ *is an* $l \times l$ *matrix whose entries belong to* $\mathscr{B}^2 S^1$*, and that*

$$p(x,\xi) + p(x,\xi)^* \geqslant 0$$

for $|\xi| \geqslant R$ *with some* $R > 0$*. Then there exists* $C_1 > 0$ *which is independent of* l*, such that for any* $u \in (\mathscr{S})^l$

$$\operatorname{Re}(p(x,D)u, u) \geqslant -C_1 A(p)\|u\|^2,$$

where

$$A(p) = \sup_{\substack{X \in \mathbb{C}^l \\ |X|=1}} |{}^t X P X|, \quad P = [\|p_{ij}\|_{\mathscr{B}^2 S^1,\nu}]_{i,j=1,\dots,l},$$

and ν *is some positive integer.*

Loosely speaking, Theorem 3, Lemma 4 and Lemma 5 allow us to deal with $p(x,\xi) \in \mathscr{B}^2 S^m$ ($m = 0, 1$) as if it belonged to S^m. Now, let us consider commutator estimates of pseudodifferential operators with constant coefficients. First we recall the Kato-Ponce commutator estimates.

Theorem 6 (Kato and Ponce [11, Lemma X1])**.** *If* $\theta > 0$*, then for any* $f, g \in \mathscr{S}$

$$\|\langle D\rangle^\theta (fg) - f\langle D\rangle^\theta g\| \leqslant C(\|\partial f\|_{L^\infty}\|g\|_{\theta-1} + \|f\|_\theta \|g\|_{L^\infty}). \tag{8}$$

Here we remark that Kato and Ponce actually proved L^p-version of (8). Next we give the Leibniz formula for Fourier multipliers.

Lemma 7. *Let* $k = 2, 3, 4, \dots$*, and let* $m \geqslant 1$ *and* $\theta > n/2 + 1$*. If* $p(\xi) \in S^m$*, then there exists a positive constant* $C_{m,\theta}$ *which is independent of* k*, such that for any* $f_1, \dots, f_k \in \mathscr{S}$

$$\|p(D)(\prod_{j=1}^k f_j)\| \leqslant C_{m,\theta}^k \sum_{\nu=1}^k \|f_\nu\|_m \prod_{\substack{j=1 \\ j\neq\nu}}^k \|f_j\|_{\theta-1}, \tag{9}$$

$$\|p(D)(\prod_{j=1}^k f_j) - \sum_{\nu=1}^k \prod_{\substack{j=1 \\ j\neq\nu}}^k f_j p(D) f_\nu\| \leqslant C_{m,\theta}^k \sum_{\nu=1}^k \|f_\nu\|_{m-1} \prod_{\substack{j=1 \\ j\neq\nu}}^k \|f_j\|_\theta. \tag{10}$$

Proof. First we show (9). We denote the Fourier transform of f by $\hat{f}$ or $\mathscr{F}[f]$, and the convolution of functions on $\mathbb{R}^n$ by $*$ respectively. Using the Plancherel-Parseval formula and the Sobolev embedding, we deduce

$$\|p(D)(\prod_{j=1}^k f_j)\|$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}^n} |p(\xi)\hat{f}_1 * \cdots * \hat{f}_k(\xi)|^2 d\xi\right)^{1/2} \\
&\leqslant C \sum_{\nu=1}^{k} \left(\int_{\mathbb{R}^n} |\hat{f}_1 * \cdots * \hat{f}_{\nu-1} * \mathscr{F}[\langle D\rangle^m f_\nu] * \hat{f}_{\nu+1} * \cdots * \hat{f}_k(\xi)|^2 d\xi\right)^{1/2} \\
&= C \sum_{\nu=1}^{k} \| \prod_{\substack{j=1 \\ j\neq\nu}}^{k} f_j \langle D\rangle^m f_\nu \| \\
&\leqslant C \sum_{\nu=1}^{k} \prod_{\substack{j=1 \\ j\neq\nu}}^{k} \|f_j\|_{L^\infty} \|f_\nu\|_m \\
&\leqslant C C_0^{k-1} \sum_{\nu=1}^{k} \prod_{\substack{j=1 \\ j\neq\nu}}^{k} \|f_j\|_{\theta-1} \|f_\nu\|_m \\
&\leqslant C_1^k \sum_{\nu=1}^{k} \prod_{\substack{j=1 \\ j\neq\nu}}^{k} \|f_j\|_{\theta-1} \|f_\nu\|_m,
\end{aligned}
$$

where $C_1 = \max\{C, C_0\}$.

Next we show (10). Set $\sigma(\xi,\eta) = p(\xi+\eta) - p(\xi) - p(\eta)$ for short. Here we claim

$$|\sigma(\xi,\eta)| \leqslant C\langle\xi\rangle^{m-1}\langle\eta\rangle \quad \text{for} \quad |\xi| \geqslant |\eta|. \tag{11}$$

Indeed, the mean value theorem implies

$$\sigma(\xi,\eta) = \sum_{j=1}^{n} \eta_j \int_0^1 \frac{\partial p}{\partial \xi_j}(\xi+\rho\eta)d\rho - p(\eta).$$

Then we have

$$|\sigma(\xi,\eta)| \leqslant C|\eta| \int_0^1 \langle \xi+\rho\eta\rangle^{m-1} d\rho + C\langle\eta\rangle^m.$$

Since $|\xi| \geqslant |\eta|$ and $m-1 \geqslant 0$, we get

$$|\sigma(\xi,\eta)| \leqslant C\langle\xi\rangle^{m-1}\langle\eta\rangle + C\langle\eta\rangle^m,$$

which is (11).

Now we show (10) for $k=2$. The Plancherel-Parseval formula gives

$$
\begin{aligned}
&\|p(D)(fg) - gp(D)f - fp(D)g\| \\
&\quad = \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \sigma(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\right|^2 d\xi\right)^{1/2}.
\end{aligned}
$$

We split the above integration in η into two pieces:

$$\int_{\mathbb{R}^n} \cdots d\eta = \int_{|\xi-\eta|\geqslant|\eta|} \cdots d\eta + \int_{|\xi-\eta|<|\eta|} \cdots d\eta.$$

Then we have

$$\|p(D)(fg) - gp(D)f - fp(D)g\| \leqslant \mathrm{I} + \mathrm{II},$$

$$\mathrm{I} = \left(\int_{\mathbb{R}^n} \left|\int_{|\xi-\eta|\geqslant|\eta|} \sigma(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\right|^2 d\xi\right)^{1/2},$$

$$\mathrm{II} = \left(\int_{\mathbb{R}^n} \left| \int_{|\xi-\eta|<|\eta|} \sigma(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta \right|^2 d\xi\right)^{1/2}.$$

On one hand, applying (11), the Young and the Schwarz inequalities in order of precedence, we deduce

$$\begin{aligned}
\mathrm{I} &\leqslant C\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \langle\xi-\eta\rangle^{m-1}|\hat{f}(\xi-\eta)|\langle\eta\rangle|\hat{g}(\eta)|d\eta \right|^2 d\xi\right)^{1/2} \\
&\leqslant C\|f\|_{m-1}\int_{\mathbb{R}^n}\langle\eta\rangle|\hat{g}(\eta)|d\eta \\
&= C\|f\|_{m-1}\int_{\mathbb{R}^n}\langle\eta\rangle^{-(\theta-1)}|\langle\eta\rangle^{\theta}\hat{g}(\eta)|d\eta \\
&\leqslant C\left(\int_{\mathbb{R}^n}\langle\eta\rangle^{-2(\theta-1)}d\eta\right)^{1/2}\|f\|_{m-1}\|g\|_\theta \\
&= C'\|f\|_{m-1}\|g\|_\theta. \qquad (12)
\end{aligned}$$

Here we used $\theta-1>n/2$. On the other hand, using (11) again, and changing variable by $\eta\mapsto\zeta=\xi-\eta$, we have

$$\mathrm{II} \leqslant C\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \langle\zeta\rangle|\hat{f}(\zeta)|\langle\xi-\zeta\rangle|\hat{g}(\xi-\zeta)|d\zeta \right|^2 d\xi\right)^{1/2},$$

which is reduced to I. Then we get (10) for $k=2$:

$$\|p(D)(fg)-gp(D)f-fp(D)g\| \leqslant C_2(\|f\|_{m-1}\|g\|_\theta+\|f\|_\theta\|g\|_{m-1}). \qquad (13)$$

Lastly, we prove (10) for $k\geqslant 3$. Set $\prod_{j=1}^0=1$. Applying (9) and (13) to the identity

$$\begin{aligned}
& p(D)(\prod_{j=1}^k f_j) - \sum_{\nu=1}^k \prod_{\substack{j=1\\ j\neq\nu}}^k f_j p(D) f_\nu \\
&= \sum_{\nu=1}^{k-1}\prod_{l=1}^{\nu-1} f_l\{p(D)(\prod_{j=\nu}^k f_j) - f_\nu p(D)(\prod_{j=\nu+1}^k f_j) - \prod_{j=\nu+1}^k f_j p(D) f_\nu\},
\end{aligned}$$

we deduce

$$\begin{aligned}
& \|p(D)(\prod_{j=1}^k f_j) - \sum_{\nu=1}^k \prod_{\substack{j=1\\ j\neq\nu}}^k f_j p(D) f_\nu\| \\
&\leqslant \sum_{\nu=1}^{k-1}\prod_{l=1}^{\nu-1}\|f_l\|_{L^\infty}\|p(D)(\prod_{j=\nu}^k f_j) - f_\nu p(D)(\prod_{j=\nu+1}^k f_j) - \prod_{j=\nu+1}^k f_j p(D) f_\nu\| \\
&\leqslant C_2\sum_{\nu=1}^{k-1}\prod_{l=1}^{\nu-1}\|f_l\|_{L^\infty}(\|f_\nu\|_{m-1}\|\prod_{j=\nu+1}^k f_j\|_\theta + \|f_\nu\|_\theta\|\prod_{j=\nu+1}^k f_j\|_{m-1}) \\
&\leqslant C_2\sum_{\nu=1}^{k-1}C_3^{\nu-1}\prod_{l=1}^{\nu-1}\|f_l\|_\theta(\|f_\nu\|_{m-1}\|\prod_{j=\nu+1}^k f_j\|_\theta + \|f_\nu\|_\theta\|\prod_{j=\nu+1}^k f_j\|_{m-1}) \\
&\leqslant C_2\sum_{\nu=1}^{k-1}C_3^{\nu-1}\prod_{l=1}^{\nu-1}\|f_l\|_\theta
\end{aligned}$$

$$\times\{(k-\nu)C_3^{k-\nu}\|f_\nu\|_{m-1}\prod_{j=\nu+1}^{k}\|f_j\|_\theta + C_4^{k-\nu-1}\sum_{p=\nu+1}^{k}\prod_{\substack{j=\nu\\ j\neq p}}^{k}\|f_j\|_\theta\|f_p\|_{m-1}\}$$

$$\leqslant C_5^k k\sum_{\nu=1}^{k}\|f_\nu\|_{m-1}\prod_{\substack{j=1\\ j\neq\nu}}^{k}\|f_j\|_\theta$$

$$\leqslant 2^k C_5^k\sum_{\nu=1}^{k}\|f_\nu\|_{m-1}\prod_{\substack{j=1\\ j\neq\nu}}^{k}\|f_j\|_\theta,$$

where $C_5=\max\{C_2,C_3,C_4\}$. This completes the proof. □

3. Summation properties

This section consists of miscellaneous lemmas needed later. In particular, we obtain some fine summmation properties related with Gevrey estimates. We start by giving the properties of exponentially decaying functions.

Lemma 8. *Let $s>0$, $\varepsilon>0$ and $\theta\in\mathbb{R}$. If $\exp(\varepsilon\langle x\rangle^{1/s})u_0\in H^\theta$, then there exists $q=q(n,\theta,\varepsilon,s)>0$ such that for any multi-index α*

$$\|x^\alpha u_0\|_\theta\leqslant\|\exp(\varepsilon\langle x\rangle^{1/s})u_0\|_\theta q^{|\alpha|+1}\alpha!^s. \tag{14}$$

Proof. By the L^2-boundedness theorem for pseudodifferential operators of order zero, we deduce

$$\begin{aligned}\|x^\alpha u_0\|_\theta &= \|\langle D\rangle^\theta x^\alpha u_0\|\\ &= \|\langle D\rangle^\theta(x^\alpha e^{-\varepsilon\langle x\rangle^{1/s}})\langle D\rangle^{-\theta}\langle D\rangle^\theta e^{\varepsilon\langle x\rangle^{1/s}}u_0\|\\ &\leqslant C\|x^\alpha e^{-\varepsilon\langle x\rangle^{1/s}}\|_{\mathscr{B}^\nu}\|e^{\varepsilon\langle x\rangle^{1/s}}u_0\|_\theta,\end{aligned} \tag{15}$$

where ν is a positive integer satisfying $\nu>|\theta|$. Set $\rho=\max\{0,1/s-1\}$. Using the Leibniz formula for $|\beta|\leqslant\nu$, we have

$$\begin{aligned}&|\partial^\beta(x^\alpha e^{-\varepsilon\langle x\rangle^{1/s}})|\\ &=\left|\sum_{\gamma\leqslant\beta}\frac{\beta!}{\gamma!(\beta-\gamma)!}\partial^\gamma x^\alpha\partial^{\beta-\gamma}e^{-\varepsilon\langle x\rangle^{1/s}}\right|\\ &\leqslant\sum_{\gamma\leqslant\beta,\alpha}\frac{\beta!}{\gamma!(\beta-\gamma)!}\frac{\alpha!}{\gamma!(\alpha-\gamma)!}\gamma!|x^{\alpha-\gamma}||\partial^{\beta-\gamma}e^{-\varepsilon\langle x\rangle^{1/s}}|\\ &\leqslant C_\nu\sum_{\gamma\leqslant\beta,\alpha}\frac{\beta!}{\gamma!(\beta-\gamma)!}\frac{\alpha!}{\gamma!(\alpha-\gamma)!}\gamma!\langle x\rangle^{|\alpha-\gamma|+(1/s-1)|\beta-\gamma|}e^{-\varepsilon\langle x\rangle^{1/s}}\\ &\leqslant C_\nu 2^{|\alpha|+\nu}\nu!\langle x\rangle^{|\alpha|+\rho\nu}e^{-\varepsilon\langle x\rangle^{1/s}}\\ &\leqslant C_\nu 2^{(1+\rho)\nu}\nu!4^{|\alpha|}\sup_{\tau>0}\tau^{\rho\nu}e^{-\varepsilon\tau^{1/s}}\sup_{t>0}t^{|\alpha|}e^{-\varepsilon t^{1/s}}\\ &=C'_{\nu,s}4^{|\alpha|}\sup_{t>0}t^{|\alpha|}e^{-\varepsilon t^{1/s}}\\ &=C'_{\nu,s}4^{|\alpha|}t^{|\alpha|}e^{-\varepsilon t^{1/s}}\Big|_{t=(s|\alpha|/\varepsilon)^s}\\ &=C'_{\nu,s}(4s^s e^{-s})^{|\alpha|}(|\alpha|^{|\alpha|}e^{-|\alpha|})^s\end{aligned}$$

$$\leqslant C'_{\nu,s}(4s^s e^{-s})^{|\alpha|}|\alpha|!^s.$$

Then there exists $q>0$ which is independent α, such that

$$\|x^\alpha e^{-\varepsilon\langle x\rangle^{1/s}}\|_{\mathscr{B}^\nu} \leqslant q^{|\alpha|+1}\alpha!^s. \tag{16}$$

The substitution of (16) into (15) gives (14). □

Next we present a lemma concerned with factorials.

Lemma 9. *For any multi-indices α, $\alpha^1,\dots,\alpha^p$ satisfying $\alpha=\alpha^1+\dots+\alpha^p$,*

$$\frac{|\alpha^1|!\cdots|\alpha^p|!}{\alpha^1!\cdots\alpha^p!} \leqslant \frac{|\alpha|!}{\alpha!}. \tag{17}$$

Proof. Let n be the dimension of α. Since

$$(x_1+\cdots+x_n)^{|\alpha|}=\prod_{j=1}^p(x_1+\cdots+x_n)^{|\alpha^j|}, \quad x=(x_1,\dots,x_n)\in\mathbb{R}^n,$$

the multinomial theorem gives

$$\sum_{|\gamma|=|\alpha|}\frac{|\alpha|!}{\gamma!}x^\gamma=\sum_{|\gamma^1|=|\alpha^1|}\cdots\sum_{|\gamma^p|=|\alpha^p|}\frac{|\alpha^1|!\cdots|\alpha^p|!}{\gamma^1!\cdots\gamma^p!}x^{\gamma^1+\cdots+\gamma^p}.$$

Operating $\partial^\alpha/\alpha!$ on the both sides of the above identity, we have

$$\frac{|\alpha|!}{\alpha!}=\sum_{\substack{\gamma^1+\cdots+\gamma^p=\alpha\\ |\gamma^1|=|\alpha^1|\\ \cdots\\ |\gamma^p|=|\alpha^p|}}\frac{|\alpha^1|!\cdots|\alpha^p|!}{\gamma^1!\cdots\gamma^p!},$$

which implies (17). □

Now we present a lemma concerned with the nonlinearity and multi-indices. This plays a crucial role in the estimate of nonlinearity. Let $\alpha=(\alpha_1,\dots,\alpha_n)$ be a multi-index. Set

$$\alpha_*=(\max\{0,\alpha_1-1\},\cdots,\max\{0,\alpha_n-1\}),$$

$$\alpha^\dagger=\prod_{j=1}^n\alpha_j^\dagger, \quad \alpha_j^\dagger=\max\{1,\alpha_j\}, \quad \alpha_*^\dagger=(\alpha_*)^\dagger.$$

Lemma 10. *Let l, p and q be integers satisfying $l\geqslant 0$ and $p,q\geqslant 2$ respectively. Set*

$$N=\sum_{k=0}^l\frac{(k+n-1)!}{k!(n-1)!},$$

which is the number of n-dimensional multi-indices satisfying $|\alpha|\leqslant l$. For any vector $(X(\alpha))_{|\alpha|\leqslant l}\in[0,\infty)^N$,

$$\begin{aligned}&\sum_{|\alpha|\leqslant l}\sum_{\substack{\alpha(1)+\cdots+\alpha(p)=\alpha\\ \beta(1)+\cdots+\beta(q)=\alpha}}\frac{(\alpha_*^\dagger)^2}{\prod_{j=1}^p\alpha(j)_*^\dagger\prod_{k=1}^q\beta(k)_*^\dagger}\prod_{j=1}^pX(\alpha(j))\prod_{k=1}^qX(\beta(k))\\ &\leqslant a^{n(p+q-2)}p^{2n}q^{2n}\left(\sum_{|\alpha|\leqslant l}X(\alpha)^2\right)^{(p+q)/2},\end{aligned} \tag{18}$$

where

$$a=\left(1+\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2}.$$

Proof. When $n=1$, there exist j and k such that $\alpha_*^\dagger \leqslant p\alpha(j)_*^\dagger, q\beta(k)_*^\dagger$. Without loss of generality, we can assume $j=p$ and $k=q$. Using the Schwarz inequality for the finite sum again and again, we have

$$\begin{aligned}
&\sum_{|\alpha|\leqslant l}\sum_{\substack{\alpha(1)+\cdots+\alpha(p)=\alpha\\ \beta(1)+\cdots+\beta(q)=\alpha}}\frac{(\alpha_*^\dagger)^2}{\prod_{j=1}^p\alpha(j)_*^\dagger\prod_{k=1}^q\beta(k)_*^\dagger}\prod_{j=1}^p X(\alpha(j))\prod_{k=1}^q X(\beta(k))\\
&\leqslant p^2q^2\sum_{\substack{\alpha(1)+\cdots+\alpha(p)\leqslant l\\ \beta(1)+\cdots+\beta(q)=\alpha(1)+\cdots+\alpha(p)}}\prod_{j=1}^{p-1}\frac{X(\alpha(j))}{\alpha(j)_*^\dagger}\prod_{k=1}^{q-1}\frac{X(\beta(k))}{\beta(k)_*^\dagger}X(\alpha(p))X(\beta(q))\\
&=p^2q^2\sum_{\alpha(1)=0}^{l}\frac{X(\alpha(1))}{\alpha(1)_*^\dagger}\cdots\sum_{\alpha(p-1)=0}^{l-\alpha(1)-\cdots-\alpha(p-2)}\frac{X(\alpha(p-1))}{\alpha(p-1)_*^\dagger}\\
&\times\sum_{\alpha(p)=0}^{l-\alpha(1)-\cdots-\alpha(p-1)}X(\alpha(p))\\
&\times\sum_{\beta(1)=0}^{\alpha(1)+\cdots+\alpha(p)}\frac{X(\beta(1))}{\beta(1)_*^\dagger}\cdots\sum_{\beta(q-1)=0}^{\alpha(1)+\cdots+\alpha(p)-\beta(1)-\cdots-\beta(q-2)}\frac{X(\beta(q-1))}{\beta(q-1)_*^\dagger}\\
&\times X(\alpha(1)+\cdots+\alpha(p)-\beta(1)-\cdots-\beta(q-1))\\
&\leqslant p^2q^2\sum_{\alpha(1)=0}^{l}\frac{X(\alpha(1))}{\alpha(1)_*^\dagger}\cdots\sum_{\alpha(p-1)=0}^{l-\alpha(1)-\cdots-\alpha(p-2)}\frac{X(\alpha(p-1))}{\alpha(p-1)_*^\dagger}\\
&\times\sum_{\beta(1)=0}^{l}\frac{X(\beta(1))}{\beta(1)_*^\dagger}\cdots\sum_{\beta(q-1)=0}^{l-\beta(1)-\cdots-\beta(q-2)}\frac{X(\beta(q-1))}{\beta(q-1)_*^\dagger}\\
&\times\sum_{\alpha(p)=\gamma\geqslant 0}^{l}X(\alpha(p))X(\alpha(1)+\cdots+\alpha(p)-\beta(1)-\cdots-\beta(q-1))\\
&\leqslant p^2q^2\sum_{\alpha(1)=0}^{l}\frac{X(\alpha(1))}{\alpha(1)_*^\dagger}\cdots\sum_{\alpha(p-1)=0}^{l-\alpha(1)-\cdots-\alpha(p-2)}\frac{X(\alpha(p-1))}{\alpha(p-1)_*^\dagger}\\
&\times\sum_{\beta(1)=0}^{l}\frac{X(\beta(1))}{\beta(1)_*^\dagger}\cdots\sum_{\beta(q-1)=0}^{l-\beta(1)-\cdots-\beta(q-2)}\frac{X(\beta(q-1))}{\beta(q-1)_*^\dagger}\\
&\times\left(\sum_{\alpha(p)=\gamma\geqslant 0}^{l}X(\alpha(p))^2\right)^{1/2}\\
&\times\left(\sum_{\alpha(p)=\gamma\geqslant 0}^{l}X(\alpha(1)+\cdots+\alpha(p)-\beta(1)-\cdots-\beta(q-1))^2\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant p^2q^2\left(\sum_{\alpha=0}^{l}\frac{X(\alpha)}{\alpha_*^\dagger}\right)^{p+q-2}\sum_{\alpha=0}^{l}X(\alpha)^2 \\
&\leqslant p^2q^2a^{p+q-2}\left(\sum_{\alpha=0}^{l}X(\alpha)^2\right)^{(p+q)/2}, \qquad (19)
\end{aligned}
$$

where we denote $\gamma=\beta(1)+\cdots+\beta(q-1)-\alpha(1)-\cdots-\alpha(p-1)$.

When $n=2$, using (19) twice, we deduce

$$
\begin{aligned}
&\sum_{|\alpha|\leqslant l}\sum_{\substack{\alpha(1)+\cdots+\alpha(p)=\alpha\\ \beta(1)+\cdots+\beta(q)=\alpha}}\frac{(\alpha_*^\dagger)^2}{\prod_{j=1}^p\alpha(j)_*^\dagger\prod_{k=1}^q\beta(k)_*^\dagger}\prod_{j=1}^pX(\alpha(j))\prod_{k=1}^qX(\beta(k)) \\
&\leqslant a^{p+q-2}p^2q^2\sum_{\alpha_2=0}^{l}\sum_{\substack{\alpha_2(1)+\cdots+\alpha_2(p)=\alpha_2\\ \beta_2(1)+\cdots+\beta_2(q)=\alpha_2}}\frac{\{(\alpha_2)_*^\dagger\}^2}{\prod_{j=1}^p\alpha_2(j)_*^\dagger\prod_{k=1}^q\beta_2(k)_*^\dagger} \\
&\quad\times\prod_{j=1}^p\left(\sum_{\alpha_1(j)\leqslant l-\alpha_2}X(\alpha_1(j),\alpha_2(j))^2\right)^{1/2} \\
&\quad\times\prod_{k=1}^q\left(\sum_{\beta_1(k)\leqslant l-\alpha_2}X(\beta_1(k),\beta_2(k))^2\right)^{1/2} \\
&\leqslant a^{2(p+q-2)}p^4q^4\left(\sum_{|\alpha|\leqslant l}X(\alpha)^2\right)^{(p+q)/2}.
\end{aligned}
$$

In the same way, we can obtain (18) for any $n\geqslant 3$. We omit the detail. □

We conclude this section by giving an estimate of matrices used for some linear systems later.

Lemma 11. *Let l be a positive integer, and let N be the same integer as in Lemma 10. Suppose that $B=[b_{\alpha,\beta}]_{|\alpha|\leqslant l,\beta\leqslant\alpha}$ is an $N\times N$ lower triangular matrix whose entries are suffixed by multi-indices. Note that $b_{\alpha,\beta}=0$ unless $\beta\leqslant\alpha$. For any $X=[X(\alpha)]_{|\alpha|\leqslant l}\in\mathbb{C}^N$,*

$$
\begin{aligned}
&|{}^tXB\bar{X}| \\
&\leqslant 2^nn!|X|^2\max_{\substack{\sigma\in S_n\\ \nu=0,1,\dots,n\\ \sigma(1)<\cdots<\sigma(\nu)\\ \sigma(\nu+1)<\cdots<\sigma(n)}} \\
&\quad\times\left(\sum_{\substack{\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(\nu)}\leqslant l-\alpha_{\sigma(\nu+1)}-\cdots-\alpha_{\sigma(n)}\\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1\\ \cdots\\ \beta_{\sigma(\nu)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1}}\max_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_\sigma(\nu+1),\dots\\ \cdots\\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\dots}}|b_{\beta,\beta-\alpha}|^2\right)^{1/2}, \qquad (20)
\end{aligned}
$$

where S_n is the n-dimensional symmetric group.

Proof. We split ${}^tXB\bar{X}$ into several pieces according to the index of the entries of B:

$$|{}^tXB\bar{X}|$$

$$
\begin{aligned}
&= \left| \sum_{|\alpha|\leqslant l} \sum_{\beta\leqslant\alpha} b_{\alpha,\beta} X(\alpha)\bar{X}(\beta) \right| \\
&= \left| \sum_{|\alpha|\leqslant l} \sum_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{\substack{\beta_{\sigma(1)}<\alpha_{\sigma(1)} \\ \cdots \\ \beta_{\sigma(\nu)}<\alpha_{\sigma(\nu)} \\ \beta_{\sigma(\nu+1)}\geqslant\alpha_{\sigma(\nu+1)}/2 \\ \cdots \\ \beta_{\sigma(n)}\geqslant\alpha_{\sigma(n)}/2}} b_{\alpha,\beta} X(\alpha)\bar{X}(\beta) \right| \\
&\leqslant 2^n n! \max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{|\alpha|\leqslant l} \sum_{\substack{\beta_{\sigma(1)}<\alpha_{\sigma(1)} \\ \cdots \\ \beta_{\sigma(\nu)}<\alpha_{\sigma(\nu)} \\ \beta_{\sigma(\nu+1)}\geqslant\alpha_{\sigma(\nu+1)}/2 \\ \cdots \\ \beta_{\sigma(n)}\geqslant\alpha_{\sigma(n)}/2}} |b_{\alpha,\beta}||X(\alpha)||X(\beta)| \\
&= 2^n n! \max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{|\alpha|\leqslant l} \sum_{\substack{|\beta|\leqslant l \\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1 \\ \cdots \\ \beta_{\sigma(\nu)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1 \\ \beta_{\sigma(\nu+1)}=2\alpha_{\sigma(\nu+1)},\cdots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\cdots}} |b_{\beta,\beta-\alpha}||X(\beta)||X(\beta-\alpha)|. \qquad (21)
\end{aligned}
$$

By the Schwarz inequality to the summation on $\beta_{\sigma(\nu+1)}, \dots, \beta_{\sigma(n)}$, (21) becomes

$$
\begin{aligned}
|{}^t X B \bar{X}| \leqslant 2^n n! &\max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{|\alpha|\leqslant l} \sum_{\substack{|\beta|\leqslant l \\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1 \\ \cdots \\ \beta_{\sigma(\nu)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1}} \\
&\times \max_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_{\sigma(\nu+1)},\cdots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)}/2,\cdots}} |b_{\beta,\beta-\alpha}| \\
&\times \left(\sum_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_{\sigma(\nu+1)},\cdots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\cdots}} |X(\beta)|^2 \right)^{1/2} \\
&\times \left(\sum_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_{\sigma(\nu+1)},\cdots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\cdots}} |X(\beta-\alpha)|^2 \right)^{1/2}. \qquad (22)
\end{aligned}
$$

If we apply the Schwarz inequality to the summation on $\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(\nu)}$ and $\beta_{\sigma(1)}, \cdots, \beta_{\sigma(\nu)}$, then (22) becomes

$$
\begin{aligned}
|{}^tXB\bar{X}| \leqslant 2^n n! & \max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{\alpha_{\sigma(\nu+1)}+\cdots+\alpha_{\sigma(n)}\leqslant l} \\
& \times \left(\sum_{\substack{\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(\nu)}\leqslant l-\alpha_{\sigma(\nu+1)}-\cdots-\alpha_{\sigma(n)} \\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1 \\ \cdots \\ \beta_{\sigma(\nu)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1}} \max_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_\sigma(\nu1),\dots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\dots}} |b_{\beta,\beta-\alpha}|^2 \right)^{1/2} \\
& \times \left(\sum_{\substack{\beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1 \\ \cdots \\ \beta_{\sigma(\nu)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1 \\ \beta_{\sigma(\nu+1)}=2\alpha_{\sigma(\nu+1)},\cdots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)}/2,\cdots}} |X(\beta)|^2 \right)^{1/2} \\
& \times \left(\sum_{\substack{\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(\nu)}\leqslant l-\alpha_{\sigma(\nu+1)}-\cdots-\alpha_{\sigma(n)} \\ \beta_{\sigma(\nu+1)}=2\alpha_{\sigma(\nu+1)},\cdots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)}/2,\cdots}} |X(\beta-\alpha)|^2 \right)^{1/2} \\
\leqslant 2^n n! |X|^2 & \max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{\alpha_{\sigma(\nu+1)}+\cdots+\alpha_{\sigma(n)}\leqslant l} \\
& \times \left(\sum_{\substack{\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(\nu)}\leqslant l-\alpha_{\sigma(\nu+1)}-\cdots-\alpha_{\sigma(n)} \\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1 \\ \cdots \\ \beta_{\sigma(\nu)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1}} \max_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_\sigma(\nu1),\dots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\dots}} |b_{\beta,\beta-\alpha}|^2 \right)^{1/2}.
\end{aligned}
$$

This completes the proof. □

4. LINEAR SYSTEMS

In this section we recall the L^2-well-posedness for some systems developed in [1]. Consider the initial value problem of the form

$$
(I_{2N}\partial_t - iE_{2N}\Delta + \sum_{k=1}^{n} B^k(t,x)\partial_k)w = g(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{23}
$$

$$
w(0,x) = w_0(x) \quad \text{in} \quad \mathbb{R}^n, \tag{24}
$$

where $w(t,x)$ is a $\mathbb{C}^{2N}$-valued unknown function of $(t,x) \in [0,T] \times \mathbb{R}^n$, I_p is $p \times p$ identity matrix,

$$E_{2N} = [I_N] \oplus [-I_N], \quad N = \sum_{j=0}^{l} \frac{(j+n-1)!}{j!(n-1)!}$$

which is the number of kinds of multi-indices of order at most l, and

$$B^k(t,x) = \begin{bmatrix} B^{k,1}(t,x) & B^{k,2}(t,x) \\ B^{k,3}(t,x) & B^{k,4}(t,x) \end{bmatrix},$$

$$B^{k,m}(t,x) = [b^{k,m}_{\alpha,\beta}(t,x)]_{|\alpha|,|\beta| \leqslant l}, \quad b^{k,m}_{\alpha,\beta}(t,x) = 0 \quad \text{unless} \quad \beta \leqslant \alpha.$$

We here assume that the Doi-type conditions, that is, there exists a nonnegative function $\phi(t,y)$ on $[0,T] \times \mathbb{R}$ such that $\phi(t,y) \in C([0,T]; \mathscr{B}^2(\mathbb{R}^2))$,

$$\sup_{t \in [0,T]} \int_{-\infty}^{+\infty} \phi(t,y) dy + \sup_{t \in [0,T]} \left| \int_{-\infty}^{+\infty} \partial_t \phi(t,y) dy \right| < +\infty, \tag{25}$$

$$\begin{aligned} &2^n n! \sum_{\substack{m=1,4 \\ k=1,\dots,n}} \max_{\substack{\sigma \in S_n \\ \nu=0,1,\dots,n \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \\ &\times \left(\sum_{\substack{\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(\nu)} \leqslant l-\alpha_{\sigma(\nu+1)}-\cdots-\alpha_{\sigma(n)} \\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1 \\ \cdots \\ \beta_{\sigma(n)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1}} \max_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_{\sigma}(\nu 1),\dots \\ \cdots \\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\dots}} |b^{k,m}_{\beta,\beta-\alpha}(t,x)|^2 \right)^{1/2} \\ &\leqslant \phi(t,x_j) \end{aligned} \tag{26}$$

for $(t,x) = (t,x_1,\dots,x_n) \in [0,T] \times \mathbb{R}^n$, $j = 1,\dots,n$. One can prove that the initial value problem (23)-(24) is L^2-well-posed by using the block-diagonalization technique in [1], and Doi's transformation in [3]. See [1] for the detail. To state the energy inequality needed later, we here introduce some pseudodifferential operators as follows:

$$\begin{aligned} \Lambda(t) &= I_{2N} + \frac{1}{2} E_{2N} \sum_{k=1}^{n} \begin{bmatrix} 0 & B^{k,2}(t,x) \\ B^{k,3}(t,x) & 0 \end{bmatrix} \partial_k (\mu^2 - \Delta)^{-1}, \\ K(t) &= [I_N k_1(t,x,D)] \oplus [I_N k_1'(t,x,D)], \\ k_1(t,x,\xi) &= e^{-p(t,x,\xi)}, \quad k_1'(t,x,\xi) = e^{p(t,x,\xi)}, \\ p(t,x,\xi) &= \sum_{k=1}^{n} \int_0^{x_k} \phi(t,y) dy \xi_k (\mu^2 + |\xi|^2)^{-1/2}. \end{aligned}$$

It is easy to see that $K(t)\Lambda(t)$ is automorphic on $(L^2)^{2N}$ provided that $\mu > 0$ is sufficiently large. More precisely, there exists a positive constants M and μ depending only on

$$\begin{aligned} &\sup_{t \in [0,T]} \|\phi(t,\cdot)\|_{\mathscr{B}^2} \\ &+ \sup_{t \in [0,T]} \int_{-\infty}^{+\infty} \phi(t,y) dy + \sup_{t \in [0,T]} \left| \int_{-\infty}^{+\infty} \partial_t \phi(t,y) dy \right| \end{aligned}$$

$$+2^n n! \sum_{\substack{m=1,4\\ k=1,\dots,n}} \max_{\substack{\sigma\in S_n\\ \nu=0,1,\dots,n\\ \sigma(1)<\dots<\sigma(\nu)\\ \sigma(\nu+1)<\dots<\sigma(n)}}$$

$$\times \left(\sum_{\substack{\alpha_{\sigma(1)}+\dots+\alpha_{\sigma(\nu)}\leqslant l-\alpha_{\sigma(\nu+1)}-\dots-\alpha_{\sigma(n)}\\ \beta_{\sigma(1)}=\alpha_{\sigma(1)},\dots,2\alpha_{\sigma(1)}-1\\ \dots\\ \beta_{\sigma(n)}=\alpha_{\sigma(\nu)},\dots,2\alpha_{\sigma(\nu)}-1}} \max_{\substack{\beta_{\sigma(\nu+1)}=2\alpha_\sigma(\nu 1),\dots\\ \dots\\ \beta_{\sigma(n)}=2\alpha_{\sigma(n)},\dots}} \|b^{k,m}_{\beta,\beta-\alpha}(t,\cdot)\|^2_{\mathscr{B}^2} \right)^{1/2}$$

such that

$$M^{-1}\|w\| \leqslant \|K(t)\Lambda(t)w\| \leqslant M\|w\|.$$

Now we state L^2-well-posedness.

Lemma 12. *Assume* (25) *and* (26). *Then, the initial value problem* (23)-(24) *is L^2-well-posed, that is, for any $w_0 \in (L^2)^{2N}$ and $g\in L^1(0,T;(L^2)^{2N})$,* (23)-(24) *has a unique solution w belonging to $C([0,T];(L^2)^{2N})$. Moreover, w satisfies*

$$\begin{aligned}\|K(t)\Lambda(t)w\|^2 = {} & \|K(t)\Lambda(t)w_0\|^2\\ & + \int_0^t 2\operatorname{Re}(Q(\tau)K(\tau)\Lambda(\tau)w(\tau), K(\tau)\Lambda(\tau)w(\tau))d\tau\\ & + \int_0^t 2\operatorname{Re}(R(\tau)K(\tau)\Lambda(\tau)w(\tau), K(\tau)\Lambda(\tau)w(\tau))d\tau\\ & + \int_0^t 2\operatorname{Re}(K(\tau)\Lambda(\tau)w(\tau), K(\tau)\Lambda(\tau)g(\tau))d\tau, \end{aligned} \tag{27}$$

for all $t \in [0,T]$, where

$$\sigma(Q(t))(x,\xi) = \sum_{j=1}^n 2\phi(t,x_j)\xi_j^2(\mu^2+|\xi|^2)^{-1/2} + i\sum_{j=1}^n B^{k,diag}(t,x)\xi_j,$$

$$B^{k,diag}(t,x) = \begin{bmatrix} B^{k,1}(t,x) & 0 \\ 0 & B^{k,4}(t,x)\end{bmatrix},$$

$$\sup_{t\in[0,T]} \|R(t)w\| \leqslant CM\|w\|.$$

Proof. The proof of Lemma 12 is basically same as that of [1, Lemma 6]. In particular, we make use of Lemma 11 to evaluate the matrices of coefficients. We here omit the detail. □

5. Nonlinear estimates

This section is devoted to estimating nonlinearity. For the sake of convenience, we use the following notation.

$$X^l_{\theta,s,r} = \left(\sum_{|\alpha|\leqslant l} \frac{r^{2|\alpha|}}{\alpha_*!^{2s}} \|J^\alpha u\|^2_\theta \right)^{1/2},$$

$$r>0, \quad \alpha_*! = \prod_{j=1}^n \max\{\alpha_j-1,0\}!, \quad (\alpha) = \prod_{j=1}^n \max\{\alpha_j,1\}.$$

First, we obtain an estimate related to the commutator $[J^\alpha, \partial_j]$.

Lemma 13. *For $u \in \mathscr{S}$ and $j = 1, \dots, n$,*

$$\left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}}{\alpha_* !^{2s}} \|J^\alpha \partial_j u\|_{\theta-1}^2 \right)^{1/2} \leqslant \sqrt{2}(1+2r) X_{\theta,s,r}^l. \tag{28}$$

Proof. A simple computation gives

$$[J^\alpha, \partial_j] = \begin{cases} -\alpha_j J^{\alpha - e_j} & \alpha_j \neq 0 \\ 0 & \alpha_j = 0 \end{cases}$$

On one hand, when $\alpha_j = 0$, we have

$$\|J^\alpha \partial_j u\|_{\theta-1} = \|\partial_j J^\alpha u\|_{\theta-1} \leqslant \|J^\alpha u\|_\theta. \tag{29}$$

On the other hand, when $\alpha \neq 0$, we have

$$\begin{aligned} \|J^\alpha \partial_j u\|_{\theta-1} &\leqslant \|\partial_j J^\alpha u\|_{\theta-1} + \alpha_j \|J^{\alpha-e_j} u\|_{\theta-1} \\ &\leqslant \|J^\alpha u\|_\theta + \alpha_j \|J^{\alpha - e_j} u\|_\theta. \end{aligned} \tag{30}$$

Substituting (29) and (30) into the left hand side of (28), we deduce

$$\begin{aligned} &\left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}}{\alpha_* !^{2s}} \|J^\alpha \partial_j u\|_{\theta-1}^2 \right)^{1/2} \\ &\leqslant \sqrt{2} \left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}}{\alpha_* !^{2s}} \|J^\alpha u\|_\theta^2 \right)^{1/2} + \sqrt{2} \left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}\alpha_j^2}{\alpha_* !^{2s}} \|J^{\alpha-e_j} u\|_\theta^2 \right)^{1/2} \\ &= \sqrt{2} \left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}}{\alpha_* !^{2s}} \|J^\alpha u\|_\theta^2 \right)^{1/2} + \sqrt{2} r \left(\sum_{|\beta| \leqslant l-1} \frac{r^{2|\beta|}(\beta_j+1)^2}{(\beta+e_j)_* !^{2s}} \|J^\beta u\|_\theta^2 \right)^{1/2} \\ &\leqslant \sqrt{2}(1+2r) X_{\theta,s,r}^l. \end{aligned}$$

Here we used $(\beta_j+1)^2/\max\{\beta_j^2, 1\} \leqslant 4$. $\square$

Secondly, we show the lower order estimates of the nonlinearity.

Lemma 14. *Let $\theta > n/2 + 2$. Set $\psi = |x|^2/4t$ and*

$$\begin{aligned} f_{\theta,\alpha} &= \langle D \rangle^\theta J^\alpha f(u, \partial u) \\ &\quad - \sum_{j=1}^n \sum_{\alpha' \leqslant \alpha} \frac{\alpha!}{\alpha'!(\alpha-\alpha')!} \\ &\quad \times \Big\{ (2it)^{|\alpha'|} \partial^{\alpha'} \frac{\partial f}{\partial v_j}(e^{-i\psi}u, \partial e^{-i\psi}\partial u) \partial_j \langle D \rangle^\theta J^{\alpha-\alpha'} u \\ &\qquad + (-1)^{|\alpha-\alpha'|} \partial^{\alpha'} \left(e^{2i\psi} \partial^{\alpha'} \frac{\partial f}{\partial \bar{v}_j}(e^{-i\psi}u, \partial e^{-i\psi}\partial u) \right) \partial_j \overline{\langle D \rangle^\theta J^{\alpha-\alpha'} u} \Big\}. \end{aligned}$$

Then, there exist a positive constant $C_{\theta,n}$ such that for any $u \in \mathscr{S}$ and $l \in \mathbb{N}$,

$$\left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}}{\alpha_* !^{2s}} \|f_{\theta,\alpha}\|^2 \right)^{1/2} \leqslant C_{\theta,n} \sum_{p=1}^\infty A_p (C_{\theta,n} X_{\theta,s,r}^{l-1})^{2p} (C_{\theta,n} X_{\theta,s,r}^l).$$

Proof. For any multi-indices $\beta, \bar\beta \in (\mathbb{N}\cup\{0\})^{n+1}$ satisfying $|\beta| = p+1$ and $|\bar\beta| = p$, and for $q = 0, 1, \dots, 2p$, set

$$\partial_{q,\beta\bar\beta} = \begin{cases} 1 & (q \leqslant \beta_0) \\ 1 & (2p+1 \leqslant q \leqslant 2p+\beta_0) \\ \partial_j & (\beta_0+\cdots+\beta_{j-1} \leqslant q \leqslant \beta_0+\cdots+\beta_j - 1) \\ \partial_j & (p+\bar\beta_0+\cdots+\bar\beta_{j-1}+1 \leqslant q \leqslant p+\bar\beta_0+\cdots+\bar\beta_j) \end{cases}$$

where $\beta = (\beta_0, \beta_1, \dots, \beta_n)$ and $\bar\beta = (\bar\beta_0, \bar\beta_1, \dots, \bar\beta_n)$. We split $f_{\theta,\alpha}$ into two parts: $f_{\theta,\alpha} = g_{\theta,\alpha} + h_{\theta,\alpha}$,

$$\begin{aligned} g_{\theta,\alpha} = &\sum_{p=1}^{\infty} \sum_{\substack{|\beta|=p+1 \\ |\bar\beta|=p}} f_{\beta\bar\beta} \sum_{\alpha^0+\cdots+\alpha^{2p}=\alpha} \frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}(-1)^{|\alpha^{p+1}+\cdots+\alpha^{2p}|} \\ &\times \Biggl\{ \langle D\rangle^\theta \left(\prod_{q=0}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \prod_{q'=p+1}^{2p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \right) \\ &\quad - \sum_{q_1=\beta_0}^{p} \langle D\rangle^\theta J^{\alpha^{q_1}}\partial_{q_1,\beta\bar\beta}u \prod_{\substack{q=0 \\ q\ne q_1}}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \prod_{q'=p+1}^{2p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \\ &\quad - \prod_{q=0}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \sum_{q_1=p+1+\bar\beta_0}^{2p} \overline{\langle D\rangle^\theta J^{\alpha^{q_1}}\partial_{q_1,\beta\bar\beta}u} \prod_{\substack{q'=p+1 \\ q'\ne q_1}}^{p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \Biggr\}, \end{aligned}$$

$$\begin{aligned} h_{\theta,\alpha} = &\sum_{p=1}^{\infty} \sum_{\substack{|\beta|=p+1 \\ |\bar\beta|=p}} f_{\beta\bar\beta} \sum_{\alpha^0+\cdots+\alpha^{2p}=\alpha} \frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}(-1)^{|\alpha^{p+1}+\cdots+\alpha^{2p}|} \\ &\times \Biggl\{ \sum_{q_1=\beta_0}^{p} \langle D\rangle^\theta [J^{\alpha^{q_1}}, \partial_{q_1,\beta\bar\beta}]u \prod_{\substack{q=0 \\ q\ne q_1}}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \prod_{q'=p+1}^{2p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \\ &\quad + \prod_{q=0}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \sum_{q_1=p+1+\bar\beta_0}^{2p} \overline{\langle D\rangle^\theta [J^{\alpha^{q_1}}, \partial_{q_1,\beta\bar\beta}]u} \prod_{\substack{q'=p+1 \\ q'\ne q_1}}^{p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \Biggr\}. \end{aligned}$$

Using Theorem 6 and Lemma 7, we deduce

$$\begin{aligned} &\Biggl\| \langle D\rangle^\theta \left(\prod_{q=0}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \prod_{q'=p+1}^{2p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \right) \\ &\quad - \sum_{q_1=\beta_0}^{p} \langle D\rangle^\theta J^{\alpha^{q_1}}\partial_{q_1,\beta\bar\beta}u \prod_{\substack{q=0 \\ q\ne q_1}}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \prod_{q'=p+1}^{2p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \\ &\quad - \prod_{q=0}^{p} J^{\alpha^q}\partial_{q,\beta\bar\beta}u \sum_{q_1=p+1+\bar\beta_0}^{2p} \overline{\langle D\rangle^\theta J^{\alpha^{q_1}}\partial_{q_1,\beta\bar\beta}u} \prod_{\substack{q'=p+1 \\ q'\ne q_1}}^{p} \overline{J^{\alpha^{q'}}\partial_{q',\beta\bar\beta}u} \Biggr\| \\ &\leqslant C_\theta^{2p+1-(\beta_0+\bar\beta_0)} (2p+1-(\beta_0+\bar\beta_0)) \prod_{q=0}^{2p} \|J^{\alpha^q}\partial_{q,\beta\bar\beta}u\|_{\theta-1} \end{aligned}$$

$$\leqslant C_\theta^{2p+1}(2p+1)\prod_{q=0}^{2p}\|J^{\alpha^q}\partial_{q,\beta\bar\beta}u\|_{\theta-1}$$

This estimate and the Minkowski inequality show that

$$\begin{aligned}
&\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|g_{\theta,\alpha}\|^2\\
&\leqslant \sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\biggl\{\sum_{p=1}^{\infty}A_p(2p+1)C_\theta^{2p+1}\sum_{\alpha^0+\alpha^{2p}=\alpha}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\prod_{q=0}^{2p}\|J^{\alpha^q}\partial_{q,\beta\bar\beta}u\|_{\theta-1}\biggr\}^2\\
&\leqslant \biggl[\sum_{p=1}^{\infty}A_p(2p+1)C_\theta^{2p+1}\biggl\{\sum_{|\alpha|\leqslant l}\\
&\times\biggl(\sum_{\alpha^0+\alpha^{2p}=\alpha}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\prod_{q=0}^{2p}\|J^{\alpha^q}\partial_{q,\beta\bar\beta}u\|_{\theta-1}\biggr)^2\biggr\}^{1/2}\biggr]^2\\
&\leqslant \biggl[\sum_{p=1}^{\infty}A_p(2p+1)C_\theta^{2p+1}\biggl\{\sum_{|\alpha|\leqslant l}\\
&\times\biggl(\sum_{\alpha^0+\alpha^{2p}=\alpha}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\biggl(\frac{\alpha_*^0!\cdots\alpha_*^{2p}!}{\alpha_*!}\biggr)^{s-1}\prod_{q=0}^{2p}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|J^{\alpha^q}\partial_{q,\beta\bar\beta}u\|_{\theta-1}\biggr)^2\biggr\}^{1/2}\biggr]^2\\
&\leqslant \biggl[\sum_{p=1}^{\infty}A_p(2p+1)C_\theta^{2p+1}\biggl\{\sum_{|\alpha|\leqslant l}\biggl(\sum_{\alpha^0+\alpha^{2p}=\alpha}\frac{(\alpha)}{(\alpha^0)\cdots(\alpha^{2p})}\prod_{q=0}^{2p}A(\alpha^q)\biggr)^2\biggr\}^{1/2}\biggr]^2,
\end{aligned}$$

where

$$A(\alpha)=\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\max\Bigl\{\|J^\alpha u\|_{\theta-1},\|J^\alpha\partial_1 u\|_{\theta-1},\dots,\|J^\alpha\partial_n u\|_{\theta-1}\Bigr\}.$$

Using Lemma 10 and (28), we deduce

$$\begin{aligned}
&\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|g_{\theta,\alpha}\|^2\\
&\leqslant\biggl[\sum_{p=1}^{\infty}A_p(2p+1)C_\theta^{2p+1}(a^n)^{2p}\biggl(\sum_{|\alpha|\leqslant l-1}A(\alpha)^2\biggr)^p\biggl(\sum_{|\alpha|\leqslant l}A(\alpha)^2\biggr)^{1/2}\biggr]^2\\
&\leqslant\biggl[\sum_{p=1}^{\infty}A_p(2p+1)C_\theta^{2p+1}(a^n)^{2p}C_r^{2p+1}(X_{\theta,s,r}^{l-1})^{2p}X_{\theta,s,r}^{l}\biggr]^2. \qquad (31)
\end{aligned}$$

In the same way, we can get

$$\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|h_{\theta,\alpha}\|^2\leqslant\biggl[\sum_{p=1}^{\infty}A_pC_{\theta,n}^{2p+1}(X_{\theta,s,r}^{l-1})^{2p}X_{\theta,s,r}^{l}\biggr]^2. \qquad (32)$$

Combining (31) and (32), we obtain Lemma 14. □

To use the linear estimates obtained in Section 4, we need the estimates of coefficient matrices of the system for ${}^t\left[r^{|\alpha|}\langle D\rangle^\theta J^\alpha u/\alpha_*!^s, r^{|\alpha|}\langle D\rangle^\theta\overline{J^\alpha u}/\alpha_*!^s\right]_{|\alpha|\leqslant l}$. For this purpose, we here define some

matrices appearing in the system as follows. For $j=1,\dots,n$ and $l\in\mathbb{N}$, we set

$$B_j^l=\begin{bmatrix} C_{j,1}^l & C_{j,2}^l \\ \overline{C_{j,2}^l} & \overline{C_{j,1}^l}\end{bmatrix} \quad C_{j,1}^l=\left[b_{j,1,\alpha\beta}^l\right]_{|\alpha|,|\beta|\leqslant l}, \quad C_{j,2}^l=\left[b_{j,2,\alpha\beta}^l\right]_{|\alpha|,|\beta|\leqslant l},$$

$$b_{j,1,\alpha\beta}^l=\begin{cases} \dfrac{\beta_*!^s r^{|\alpha-\beta|}}{\alpha_*!^s}\displaystyle\sum_{p=1}^{\infty}\sum_{|\gamma|-1=|\bar\gamma|=p} f_{\gamma\bar\gamma}\gamma_j \\ \quad\times\displaystyle\sum_{j,1}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!\beta!}(-1)^{|\alpha^{p+1}+\cdots+\alpha^{2p}|} \\ \quad\times\displaystyle\prod_{\substack{q=0\\ q\ne\gamma_1+\cdots+\gamma_{j-1}}}^{p} J^{\alpha^q}\partial_{q,\gamma\bar\gamma}u\prod_{q'=p+1}^{2p}\overline{J^{\alpha^{q'}}\partial_{q',\gamma\bar\gamma}u} & \text{if} \quad \beta\leqslant\alpha, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sum_{j,1}=\sum_{\alpha^0+\cdots+\alpha^{m-1}+\alpha^{m+1}\cdots+\alpha^{2p}=\alpha-\beta}, \quad m=\gamma_0+\cdots+\gamma_{j-1},$$

$$b_{j,2,\alpha\beta}^l=\begin{cases} \dfrac{\beta_*!^s r^{|\alpha-\beta|}}{\alpha_*!^s}\displaystyle\sum_{p=1}^{\infty}\sum_{|\gamma|-1=|\bar\gamma|=p} f_{\gamma\bar\gamma}\bar\gamma_j \\ \quad\times\displaystyle\sum_{j,2}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!\beta!}(-1)^{|\alpha^{p+1}+\cdots+\alpha^{2p}|} \\ \quad\times\displaystyle\prod_{q=0}^{p} J^{\alpha^q}\partial_{q,\gamma\bar\gamma}u\prod_{\substack{q'=p+1\\ q'\ne p+\bar\gamma_0+\cdots+\bar\gamma_{j-1}+1}}^{2p}\overline{J^{\alpha^{q'}}\partial_{q',\gamma\bar\gamma}u} & \text{if} \quad \beta\leqslant\alpha, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sum_{j,2}=\sum_{\alpha^0+\cdots+\alpha^{m-1}+\alpha^{m+1}\cdots+\alpha^{2p}=\alpha-\beta}, \quad m=p+\bar\gamma_0+\cdots+\bar\gamma_{j-1}.$$

We need the estimates of the above matrices later.

Lemma 15. *Let $\theta>n/2+3$. Then, there exists a positive constant $C_{\theta,n}$ which is independent of $l\in\mathbb{N}$, such that for $j=1,\dots,n$ and for any $u\in C^1([0,T];\mathscr{S})$ solving* (1)*,*

$$\|B_j^l(t)\|_{\mathscr{B}^2}\leqslant C_{\theta,n}\sum_{p=1}^{\infty}A_p(C_{\theta,n}X_{\theta,s,r}^{l-1}(t))^{2p-1}(C_{\theta,n}X_{\theta,s,r}^l(t)), \tag{33}$$

$$\begin{aligned}\|\partial_t B_l^l(t)\|_{\mathscr{B}^0}\leqslant{}& C_{\theta,n}\sum_{p=1}^{\infty}A_p(C_{\theta,n}X_{\theta,s,r}^{l-1}(t))^{2p-1}(C_{\theta,n}X_{\theta,s,r}^l(t)) \\ &\times\left(1+\sum_{q=1}^{\infty}(C_{\theta,n}X_{\theta,s,r}^{l-1}(t))^{2q}\right),\end{aligned} \tag{34}$$

Proof. Simple computation shows that

$$\begin{aligned}\|B_j^l(t)\|_{\mathscr{B}^2}&\leqslant 2\sum_{k=1,2}\|C_{j,k}^l(t)\|_{\mathscr{B}^2} \\ &\leqslant 2\sum_{k=1,2}\left|\left[\|b_{j,k,\alpha\beta}^l(t)\|_{\mathscr{B}^2}\right]_{|\alpha|,|\beta|\leqslant l}\right|\end{aligned}$$

$$= \sum_{k=1,2} I_k(t). \tag{35}$$

We show that $I_1(t)$ bounded by the right hand side of (33). For the sake of convenience, set

$$A(\alpha) = \begin{cases} \max\Big\{\|J^\alpha u\|, \|J^\alpha \partial_1 u\|, \dots, \|J^\alpha \partial_n u\|\Big\} & \text{if} \quad |\alpha| \leqslant l-1, \\ 0 & \text{if} \quad |\alpha| = l. \end{cases}$$

Using (9), we have for any $\beta \leqslant \alpha$

$$\begin{aligned} \|b_{j,1,\alpha(\alpha-\beta)}\|_{\theta-1} \leqslant{}& \frac{(\alpha-\beta)_*!^s r^{|\beta|}}{\alpha_*!^s} C_{\theta,n} \sum_{p=1}^{\infty} A_p p^2 C_{\theta,n}^{2p} \\ & \times \sum_{\alpha^1+\cdots+\alpha^{2p}=\beta} \frac{\alpha!}{\alpha^1! \cdots \alpha^{2p}!} \prod_{q=1}^{2p} A(\alpha^q) \\ \leqslant{}& C_{\theta,n} \sum_{p=1}^{\infty} A_p p^2 C_{\theta,n}^{2p} \\ & \times \sum_{\alpha^1+\cdots+\alpha^{2p}=\beta} \frac{(\alpha-\beta)_*!^{s-1} \alpha_*^1!^{s-1} \cdots \alpha_*^{2p}!^{s-1}}{\alpha_*!^{s-1}} \\ & \times \frac{(\alpha)}{(\alpha^1)\cdots(\alpha^{2p})((\alpha-\beta))} \prod_{q=1}^{2p} A(\alpha^q) \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \\ \leqslant{}& C_{\theta,n} \frac{(\alpha)}{(\beta)((\alpha-\beta))} \sum_{p=1}^{\infty} A_p p^2 C_{\theta,n}^{2p} \frac{(\beta)}{(\alpha^1)\cdots(\alpha^{2p})} \prod_{q=1}^{2p} A(\alpha^q) \frac{r^{|\alpha^q|}}{\alpha_*^q!^s}. \end{aligned}$$

Using (20) and the above estimates, we deduce

$$\begin{aligned} I_1(t) \leqslant{}& 2^n n! \max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n-1 \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} \sum_{\beta_{\sigma(\nu+1)}+\cdots+\beta_{\sigma(n)}\leqslant l} \\ & \times \Bigg\{ \sum_{\substack{\beta_{\sigma(1)} \\ \vdots \\ \beta_{\sigma(\nu)}}} \sum_{\substack{\alpha_{\sigma(1)}<2\beta_{\sigma(1)} \\ \vdots \\ \alpha_{\sigma(\nu)}<2\beta_{\sigma(\nu)}}} \max_{\substack{\alpha_{\sigma(\nu+1)}\geqslant 2\beta_{\sigma(\nu1)} \\ \vdots \\ \alpha_{\sigma(n)}\geqslant 2\beta_{\sigma(n)}}} \frac{(\alpha)^2}{(\beta)^2((\alpha-\beta))^2} \\ & \times \left(\sum_{p=1}^{\infty} A_p p^2 C_{\theta,n}^{2p} \sum_{\alpha^1+\cdots+\alpha^{2p}=\beta} \frac{(\beta)}{(\alpha^1)\cdots(\alpha^{2p})} \prod_{q=1}^{2p} A(\alpha^q) \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^2 \Bigg\}^{1/2}. \end{aligned} \tag{36}$$

We here remark that if

$$\alpha_{\sigma(1)} < 2\beta_{\sigma(1)}, \dots, \alpha_{\sigma(\nu)} < 2\beta_{\sigma(\nu)}, \quad \alpha_{\sigma(\nu+1)} \geqslant 2\beta_{\sigma(\nu+1)}, \dots, \alpha_{\sigma(n)} \geqslant 2\beta_{\sigma(n)},$$

then

$$\begin{aligned} \frac{(\alpha)}{(\beta)((\alpha-\beta))} ={}& \frac{(\alpha_{\sigma(1)})}{(\beta_{\sigma(1)})} \cdots \frac{(\alpha_{\sigma(\nu)})}{(\beta_{\sigma(\nu)})} \frac{(\alpha_{\sigma(\nu+1)})}{((\alpha-\beta)_{\sigma(\nu+1)})} \cdots \frac{(\alpha_{\sigma(n)})}{((\alpha-\beta)_{\sigma(n)})} \\ & \times \frac{1}{((\alpha-\beta)_{\sigma(1)})\cdots((\alpha-\beta)_{\sigma(\nu)})} \frac{1}{(\beta_{\sigma(\nu+1)})\cdots(\beta_{\sigma(n)})} \end{aligned}$$

$$\leqslant \frac{2^n}{((\alpha-\beta)_{\sigma(1)})\cdots((\alpha-\beta)_{\sigma(\nu)})(\beta_{\sigma(\nu+1)})\cdots(\beta_{\sigma(n)})}.$$

Substituting this into (36) and using the Schwarz inequality to the summation on $\beta_{\sigma(\nu+1)},\dots,\beta_{\sigma(n)}$, we deduce

$$\begin{aligned}
I_1(t) \leqslant 2^{2n}n!^2 & \max_{\substack{\sigma\in S_n \\ \nu=0,1,\dots,n-1 \\ \sigma(1)<\cdots<\sigma(\nu) \\ \sigma(\nu+1)<\cdots<\sigma(n)}} 2^\nu \sum_{\beta_{\sigma(\nu+1)}+\cdots+\beta_{\sigma(n)}\leqslant l} \frac{1}{(\beta_{\sigma(\nu+1)})\cdots(\beta_{\sigma(n)})} \\
& \times \Biggl\{ \sum_{\beta_{\sigma(1)}} \sum_{\alpha_{\sigma(1)}<2\beta_{\sigma(1)}} \\
& \qquad \vdots \qquad\quad \vdots \\
& \quad \sum_{\beta_{\sigma(\nu)}} \sum_{\alpha_{\sigma(\nu)}<2\beta_{\sigma(\nu)}} \\
& \times \left(\sum_{p=1}^\infty A_p p^2 C_{\theta,n}^{2p} \sum_{\alpha^1+\cdots+\alpha^{2p}=\beta} \frac{(\beta)}{(\alpha^1)\cdots(\alpha^{2p})} \prod_{q=1}^{2p} A(\alpha^q) \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^2 \Biggr\}^{1/2}. \\
& \leqslant 2^{2n}n!^2 a^n \Biggl\{ \sum_{|\beta|\leqslant l} \\
& \times \left(\sum_{p=1}^\infty A_p p^2 C_{\theta,n}^{2p} \sum_{\alpha^1+\cdots+\alpha^{2p}=\beta} \frac{(\beta)}{(\alpha^1)\cdots(\alpha^{2p})} \prod_{q=1}^{2p} A(\alpha^q) \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^2 \Biggr\}^{1/2}.
\end{aligned}$$

The Minkowski inequality shows that

$$\begin{aligned}
I_1(t) \leqslant{}& 2^{2n}n!^2 a^n \sum_{p=1}^\infty A_p p^2 C_{\theta,n}^{2p} \\
& \times \left\{ \sum_{|\beta|\leqslant l} \left(\sum_{\alpha^1+\cdots+\alpha^{2p}=\beta} \frac{(\beta)}{(\alpha^1)\cdots(\alpha^{2p})} \prod_{q=1}^{2p} A(\alpha^q) \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^2 \right\}^{1/2}.
\end{aligned}$$

Applying (18) to this, we have

$$\begin{aligned}
I_1(t) \leqslant{}& 2^{2n}n!^2 a^n \sum_{p=1}^\infty A_p p^{2n+2} C_{\theta,n}^{2p} (a^n)^{2p} \\
& \times \left(\sum_{|\alpha|\leqslant l-1} A(\alpha)^2 \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^{(2p-1)/2} \left(\sum_{|\alpha|\leqslant l} A(\alpha)^2 \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^{1/2}.
\end{aligned} \tag{37}$$

Since $A(\alpha)=0$ for $|\alpha|=l$, and $p^{2n+2}\leqslant e^p(2n+2)!$, (37) is bounded by

$$I_1(t) \leqslant e^2 2^{2n} n!^2 a^n (2n+2)! \sum_{p=1}^n A_p \left\{ eC_{\theta,n} a^n \left(\sum_{|\alpha|\leqslant l-1} A(\alpha)^2 \frac{r^{|\alpha^q|}}{\alpha_*^q!^s} \right)^{1/2} \right\}^{2p}.$$

Using (28) for $r\leqslant 1$, we have

$$I_1(t) \leqslant e^2 2^{2n} n!^2 a^n (2n+2)! \sum_{p=1}^n A_p \Bigl(eC_{\theta,n} a^n X_{\theta,s,r}^{l-1}(t) \Bigr)^{2p}.$$

The estimates of $I_2(t)$, (33) and (34) can be obtained similarly. If u solves (1), then

$$\partial_t J^\alpha \partial_j u = i\Delta J^\alpha \partial_j u + J^\alpha \partial_j f(u,\partial u).$$

Applying this formula to the time-derivatives of the matrices, we can show (34) in the same way as (33). We here omit the detail. □

6. UNIFORM ENERGY ESTIMATES

In this section we show that $\{X^l_{\theta,s,r}(t)\}_{l=0,1,2,\dots}$ is bounded in $C[-T,T]$. If this is true, then there exists a constant $C_0>0$ such that

$$X^\infty_{\theta,s,r}(t) = \left(\sum_\alpha \frac{r^{2|\alpha|}}{\alpha_*!^s}\|J^\alpha u(t)\|^2_\theta\right)^{1/2} \leqslant C_0 \tag{38}$$

for $t\in[-T,T]$. Let $u{\in}C([-T,T];H^\theta)$ be a solution to (1)-(2) with $e^{\varepsilon\langle x\rangle^{1/s}}u_0{\in}H^\theta$. Theorem 1 shows that $X^l_{\theta,s,r}(t)$ is well-defined for any $l=0,1,2,\dots$. Lemma 8 implies that there exist positive constants M and r such that

$$X^\infty_{\theta,s,r}(0) = \left(\sum_\alpha \frac{r^{2|\alpha|}}{\alpha_*!^s}\|x^\alpha u_0\|^2_\theta\right)^{1/2} \leqslant M. \tag{39}$$

Without loss of generality, we may assume $r\leqslant 1$. Since the finite sum $X^l_{\theta,s,r}(t)$ is well-defined, it suffices to prove (38) for small $T>0$. In order to make use of the energy estimates in Section 4, we here define functions and pseudodifferential operators:

$$w^l = {}^t\left[\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^\theta J^\alpha u, \frac{r^{|\alpha|}}{\alpha_*!^s}\overline{\langle D\rangle^\theta J^\alpha u}\right]_{|\alpha|\leqslant l}, \quad g^l = {}^t\left[\frac{r^{|\alpha|}}{\alpha_*!^s}f_{\theta,\alpha}, \frac{r^{|\alpha|}}{\alpha_*!^s}\overline{f_{\theta,\alpha}}\right]_{|\alpha|\leqslant l},$$

$$k^l_1(t,x,\xi) = \exp\left(A\sum_{j=1}^n \xi_j(\nu^2+|\xi|^2)^{-1/2}\int_{-\infty}^{x_j}\phi^l(t,s)ds\right),$$

$$\begin{aligned}\phi^l(t,s) = {} & \sum_{j=1}^n\sum_{|\alpha|\leqslant l}\frac{r^{|\alpha|}}{\alpha_*!^s}\int\cdots\int_{\mathbb{R}^{n-1}} \\ & \times|\langle D\rangle^\delta J^\alpha u(t,x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_n)|^2 \\ & \times dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n,\end{aligned}$$

$$\delta = \frac{\theta}{2}+\frac{n}{4}-1 > \frac{n+1}{2},$$

where A and ν are positive constans determined later,

$$N(l) = \sum_{m=0}^l \frac{(l+n-1)!}{l!(n-1)!},$$

$$k^l(t,x,\xi) = [I_{N(l)}k^l_1(t,x,\xi)]\oplus[I_{N(l)}k^l_1(t,x,\xi)^{-1}],$$
$$k^l_{\mathrm{inv}}(t,x,\xi) = k^l(t,x,\xi)^{-1},$$
$$K^l(t) = k^l(t,x,D), \quad K^l_{\mathrm{inv}}(t) = k^l_{\mathrm{inv}}(t,x,D),$$
$$\tilde\Lambda^l(t) = \frac{1}{2}\sum_{j=1}^n E_{2N(l)}\Big(B^l_j(t,x)-B^{l,\mathrm{diag}}_j(t,x)\Big)\partial_j(\nu^2-\Delta)^{-1},$$
$$\Lambda^l(t) = I_{2N(l)} - i\tilde\Lambda^l(t), \quad \Lambda^l_{\mathrm{inv}}(t) = I_{2N(l)} + i\tilde\Lambda^l(t).$$

First we determine A and ν. On one hand, in the same way as the proof of Lemma 15, we have

$$\sum_{j=1}^{n}|B_j^l(t,x)| \leqslant C_{\theta,n}^2\phi^l(t,x_m)\sum_{p=1}^{\infty}A_p\Big(C_{\theta,n}X_{\theta-2,s,r}^{l-1}(t)\Big)^{2p-1}$$

for $(t,x)\in[-T,T]\times\mathbb{R}^n$ and $m=1,\dots,n$. Hence, if $X_{\theta-2,s,r}^l(t)\leqslant 4M$, then there exists a positive constant A depending only on M, θ, n and $\{A_p\}_{p=1,2,3\dots}$ such that $|B_j^l(t,x)|\leqslant A\phi^l(t,x_m)$ for $(t,x)\in[-T,T]\times\mathbb{R}^n$ and $m=1,\dots,n$. On the other hand, it is easy to see that

$$K^l(t)\Lambda^l(t)\Lambda_{\text{inv}}^l(t)K_{\text{inv}}^l(t)=I_{N(l)}+R_1^l(t),$$
$$\Lambda_{\text{inv}}^l(t)K_{\text{inv}}^l(t)K^l(t)\Lambda^l(t)=I_{N(l)}+R_2^l(t).$$

We here remark that $R_1^l(t)$ and $R_2^l(t)$ are pseudodifferential operators of order -1 and

$$\|R_1^l(t))\|,\|R_2^l(t)\|=\mathcal{O}(\nu^{-1}).$$

Nagase's theorem shows that if $X_{\theta-1,s,r}^l(t)\leqslant 2M$, then there exist $\nu_0\geqslant 1$ and $C_M>0$ which are independent of l, such that $K^l(t)\Lambda^l(t)$ is invertible, and

$$\|K^l(t)\Lambda^l(t)\|,\|(K^l(t)\Lambda^l(t))^{-1}\|\leqslant C_M \tag{40}$$

for $\nu\geqslant\nu_0$. Set $\nu=\nu_0$ below.

Now we begin the proof of (38) for some small $T>0$. Without loss of generality, we may assume that

$$\|K^l(0)\Lambda^l(0)w^l(0)\|\leqslant M$$

for $l=0,1,2,\dots$. It suffices to consider only the forward direction in time. Let T_l be a positive time defined by

$$T_l=\sup\Big\{T>0\ \Big|\ X_{\theta-1,s,r}^l(t)^2+\|K^l(t)\Lambda^l(t)w^l(t)\|^2\leqslant 4M^2\quad\text{for}\quad t\in[0,T]\Big\}.$$

We remark that (40) is valid for $t\in[0,T_l]$. Using the Schwarz inequality, we have

$$\begin{aligned}
\frac{d}{dt}X_{\theta-1,s,r}^l(t)^2 &= 2\operatorname{Re}\sum_{|\alpha|\leqslant l}\left(\frac{r^{|\alpha|}}{\alpha_*!^s}\partial_t\langle D\rangle^{\theta-1}J^\alpha u,\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^{\theta-1}J^\alpha u,\right)\\
&= 2\operatorname{Re}\sum_{|\alpha|\leqslant l}\left(\frac{r^{|\alpha|}}{\alpha_*!^s}i\Delta\langle D\rangle^{\theta-1}J^\alpha u,\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^{\theta-1}J^\alpha u,\right)\\
&\quad+2\operatorname{Re}\sum_{|\alpha|\leqslant l}\left(\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^{\theta-1}J^\alpha f,\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^{\theta-1}J^\alpha u,\right)\\
&= 2\operatorname{Re}\sum_{|\alpha|\leqslant l}\left(\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^{\theta-1}J^\alpha f,\frac{r^{|\alpha|}}{\alpha_*!^s}\langle D\rangle^{\theta-1}J^\alpha u,\right)\\
&\leqslant 2\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|\langle D^{\theta-1}\rangle^{\theta-1}J^\alpha f\|\|J^\alpha u\|_{\theta-1}\\
&\leqslant 2\left(\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|\langle D\rangle^{\theta-1}J^\alpha f(t)\|^2\right)^{1/2}X_{\theta,s,r,}^l(t).
\end{aligned}\tag{41}$$

In the same way as Lemma 14, we can get

$$\begin{aligned}
&\left(\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|\langle D\rangle^{\theta-1}J^\alpha f(t)\|^2\right)^{1/2} X^l_{\theta,s,r,}(t)\\
&\leqslant C_{\theta,n}\sum_{p=1}^{\infty}A_pC_{\theta,n}^{2p+1}\Big(X^l_{\theta,s,r}(t)\Big)^{2p+2}\\
&\leqslant C_{\theta,n}\sum_{p=1}^{\infty}A_pC_{\theta,n}^{2p+1}C_M^{2p+2}\|K^l(t)\Lambda^l(t)w^l(t)\|^{2p+2}\\
&\leqslant C_{\theta,n}\sum_{p=1}^{\infty}A_pC_{\theta,n}^{2p+1}C_M^{2p+2}M^2p\|K^l(t)\Lambda^l(t)w^l(t)\|^{2}.
\end{aligned}$$

For $R=2C_{\theta,n}C_MM$, there exists a positive constant C_R such that

$$\left(\sum_{|\alpha|\leqslant l}\frac{r^{2|\alpha|}}{\alpha_*!^{2s}}\|\langle D\rangle^{\theta-1}J^\alpha f(t)\|^2\right)^{1/2} X^l_{\theta,s,r,}(t)\leqslant C_{\theta,n}C_R\|K^l(t)\Lambda^l(t)w^l(t)\|^2.$$

Substituting this into (41), we obtain

$$\frac{d}{dt}X^l_{\theta-1,s,r}(t)^2\leqslant 2C_{\theta,n}C_R\|K^l(t)\Lambda^l(t)w^l(t)\|^2. \tag{42}$$

On the other hand, w^l solves

$$\left(I_{2N(l)}\partial_t-i\Delta E_{2N(l)}+\sum_{j=1}^{n}B^l_j(t,x)\partial_j\right)w^l=g^l.$$

By using Lemma 12, we have

$$\begin{aligned}
\frac{d}{dt}\|K^l(t)\Lambda^l(t)w^l(t)\|^2&\leqslant 2\mathcal{C}^l(t)\|K^l(t)\Lambda^l(t)w^l(t)\|^2\\
&\quad+2\|K^l(t)\Lambda^l(t)w^l(t)\|\|K^l(t)\Lambda^l(t)g^l(t)\|,
\end{aligned}$$

where $\mathcal{C}^l(t)$ depends only on

$$\int_{\mathbb{R}}\phi^l(t,s)ds,\quad \|\phi^l(t)\|_{\mathscr{B}^2},\quad \sup_{z\in\mathbb{R}}\left|\int_{-\infty}^{z}\partial_t\phi^l(t,x)ds\right|,$$
$$\sum_{j=1}^{n}\Big(\|B^l_j(t)\|_{\mathscr{B}^2}+\|\partial_tB^l_j(t)\|_{\mathscr{B}^0}\Big).$$

It is easy to see that

$$\int_{\mathbb{R}}\phi^l(t,s)ds,\quad \|\phi^l(t)\|_{\mathscr{B}^2}\leqslant X^l_{\theta,s,r}(t). \tag{43}$$

Using

$$\partial_t\langle D\rangle^\delta J^\alpha u=i\Delta\langle D\rangle^\delta J^\alpha u+\langle D\rangle^\delta J^\alpha f,$$

and the integration by parts, we deduce

$$\begin{aligned}
\sup_{z\in\mathbb{R}}\left|\int_{-\infty}^{z}\partial_t\phi^l(t,x)ds\right|&\leqslant C\{\theta,n\}X^l_{\theta,s,r}(t)\\
&\quad\times\left(1+\sum_{p=1}^{\infty}A_pC_{\theta,n}^{2p}\Big(X^l_{\theta,s,r}(t)\Big)^{2p}\right).
\end{aligned}\tag{44}$$

(43), (44) and Lemma 15 show that there exists a positive constant D_M which depends only on M, θ and n, and is independent of l, such that $\mathcal{C}^l(t) \leqslant D_M$ for $t \in [0, T_l]$. On the other hand, Lemma 14 shows that

$$\begin{aligned}
\|K^l(t)\Lambda^l(t)g^l(t)\| &\leqslant C_M \left(\sum_{|\alpha| \leqslant l} \frac{r^{2|\alpha|}}{\alpha_*!^{2s}} \|f_{\theta,\alpha}(t)\|^2 \right)^{1/2} \\
&\leqslant C_M C_{\theta,n} \sum_{p=1}^{\infty} A_p C_{\theta,n}^{2p+1} \left(X_{\theta,s,r}^l(t) \right)^{2p+1} \\
&\leqslant C_M C_{\theta,n} C_R \|K^l(t)\Lambda^l(t)w^l(t)\|.
\end{aligned}$$

Hence, we have

$$\frac{d}{dt}\|K^l(t)\Lambda^l(t)w^l(t)\|^2 \leqslant 2(D_M + C_M C_{\theta,n} C_R)\|K^l(t)\Lambda^l(t)w^l(t)\|^2. \tag{45}$$

Combining (42) and (45), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{ X_{\theta-1,s,r}^l(t)^2 + \|K^l(t)\Lambda^l(t)w^l(t)\|^2 \right\} \\
\leqslant & 2C_1 \left\{ X_{\theta-1,s,r}^l(t)^2 + \|K^l(t)\Lambda^l(t)w^l(t)\|^2 \right\},
\end{aligned} \tag{46}$$

where C_1 depends only on M, θ and n, and is independent of l. Integrating (46) over $[0, T_l]$, we obtain $4M^2 \leqslant 2M^2 \exp(2C_1 T_l)$, which implies that $T_l \geqslant \log 2/2C_1 > 0$. Set $T^* = \log 2/2C_1$ for short. For all l, we obtain

$$X_{\theta-1,s,r}^l(t), \|K^l(t)\Lambda^l(t)w^l(t)\| \leqslant 2M$$

for $t \in [0, T^*]$. Hence, (40) shows that

$$\begin{aligned}
X_{\theta,s,r}^l(t) &= \frac{1}{2}\|w^l(t)\| \\
&= \frac{1}{2}\|(K^l(t)\Lambda^l(t))^{-1}K^l(t)\Lambda^l(t)w^l(t)\| \\
&\leqslant \frac{C_M}{2}\|K^l(t)\Lambda^l(t)w^l(t)\| \\
&\leqslant C_M M
\end{aligned}$$

for $t \in [0, T^*]$. Thus we obtain $X_{\theta,s,r,}^{\infty}(t) \leqslant C_M M$ for $t \in [0, T^*]$. This completes the proof of the uniform energy estimates.

7. Gevrey estimates of solutions

In this section we complete the proof of Theorem 2. For $k \in \mathbb{N} \cup \{0\}$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_1)$, set $k_+ = \max\{k, 1\}$ and $\alpha_+ = ((\alpha_1)_+, \dots, (\alpha_n)_+)$. In the previous section, we have proved that

$$\sum_{\alpha} \frac{r^{2|\alpha|}}{\alpha_*!^{2s}} \|J^\alpha u(t)\|_\theta^2 \leqslant M^2$$

for $t \in [-T, T]$ with some positive constants r, M and $s \geqslant 1$. The Schwartz inequality shows that

$$\begin{aligned}
\sum_{\alpha} \frac{r^{|\alpha|}}{\alpha!^s}\|J^\alpha u(t)\|_\theta &= \sum_{\alpha} \frac{1}{\alpha_+^s} \frac{r^{|\alpha|}}{\alpha_*!^s}\|J^\alpha u(t)\|_\theta \\
&\leqslant \left(\sum_{\alpha} \frac{1}{\alpha_+^{2s}} \right)^{1/2} \left(\sum_{\alpha} \frac{r^{2|\alpha|}}{\alpha_*!^{2s}} \|J^\alpha u(t)\|_\theta^2 \right)^{1/2} \\
&\leqslant a^n M
\end{aligned} \tag{47}$$

for $t \in [-T, T]$. In order to obtain (5) from (47), we need two lemmas.

Lemma 16. *Suppose that $s \geqslant 1/2$ and*

$$\sum_{\alpha} \frac{r^{|\alpha|}}{\alpha!^s} \|J^\alpha u(t)\|_\theta \leqslant M_0$$

for $t \in [-T, T] \setminus \{0\}$. Then there here exist positive constants ρ and M_1 such that

$$\sum_{\alpha} \frac{1}{|\alpha|!^s} \left(\frac{|t|}{\rho}\right)^{|\alpha|} \|\langle x\rangle^{-|\alpha|} \partial^\alpha u(t)\|_\theta \leqslant M_1$$

for $t \in [-T, T] \setminus \{0\}$.

Lemma 17. *For any smooth function u of (t, x),*

$$\begin{aligned} &\|\langle x\rangle^{-|\alpha|-2m} \partial_t^m \partial^{\alpha+e_j} u(t)\|_{\theta-1} \\ &\quad \leqslant C_0(|\alpha| + 2m)_+ \|\langle x\rangle^{-|\alpha|-2m} \partial_t^m \partial^\alpha u(t)\|_\theta, \end{aligned} \tag{48}$$

$$\begin{aligned} &\|\langle x\rangle^{-2} \partial_j \{\langle x\rangle^{-|\alpha|-2m} \partial_t^m \partial^{\alpha+e_k} u(t)\}\|_{\theta-1} \\ &\quad \leqslant \|\langle x\rangle^{-|\alpha+e_j+e_k|-2m} \partial_t^m \partial^{\alpha+e_j+e_k} u(t)\|_{\theta-1} \\ &\quad + C_0(|\alpha| + 2m)_+ \|\langle x\rangle^{-|\alpha+e_k|-2m} \partial_t^m \partial^{\alpha+e_k} u(t)\|_{\theta-1}, \end{aligned} \tag{49}$$

where $C_0 > 0$ is independent of α and m.

Proof of Lemma 16. Recall the explicit formula of the hermitian polynomial

$$e^{-a\tau^2} \left(\frac{d}{d\tau}\right)^\mu e^{a\tau^2} = \sum_{\nu \leqslant \mu/2} \frac{\mu!}{\nu!(\mu - 2\nu)!} a^{\mu-\nu} (2\tau)^{\mu-2\nu}$$

for $a, \tau \in \mathbb{R}$ and $\mu \in \mathbb{N}$. Applying this to the definition of the operator J, we deduce

$$\begin{aligned} \partial^\alpha u &= \partial^\alpha \Big\{ e^{i|x|^2/4t} (e^{-i|x|^2/4t} u) \Big\} \\ &= \sum_{\beta \leqslant \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \Big(e^{-i|x|^2/4t} \partial^\beta e^{i|x|^2/4t} \Big) \Big\{ e^{i|x|^2/4t} \partial^{\alpha-\beta} \Big(e^{-i|x|^2/4t} u \Big) \Big\} \\ &= \sum_{\beta \leqslant \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \left(\frac{1}{2it}\right)^{|\alpha-\beta|} J^{\alpha-\beta} u \\ &\qquad \times \sum_{\gamma \leqslant \beta/2} \frac{\beta!}{\gamma!(\beta - 2\gamma)!} \left(\frac{i}{4t}\right)^{|\beta-\gamma|} (2x)^{\beta-2\gamma}. \end{aligned}$$

Set $\Theta = [\theta] + 1$ and $\rho_0 = \max\{4, 2(1 + T), 1/r^2\}$. Here we remark that $1 - s \leqslant 1/2$ since $s \geqslant 1/2$. We deduce

$$\begin{aligned} &\frac{1}{(|\alpha| + 2\Theta)!^s} \left(\frac{|t|}{\rho_0}\right)^{|\alpha|} \|\langle x\rangle^{-|\alpha|} \partial^\alpha u\|_\theta \\ &\leqslant \frac{1}{\alpha!^s |\alpha|^{2\Theta} \rho_0^{|\alpha|}} 2^{-|\alpha|} \sum_{\beta \leqslant \alpha} \sum_{\gamma \leqslant \beta/2} \\ &\quad \times |t|^{|\gamma|} \frac{\alpha!}{\beta!(\alpha - \beta)!} \frac{\beta!}{\gamma!(\beta - 2\gamma)!} \left\| \frac{x^{\beta-2\gamma}}{\langle x\rangle^{|\alpha|}} J^{\alpha-\beta} u \right\|_\theta \\ &= \frac{2^{-|\alpha|/2}}{|\alpha|^{2\Theta} \rho_0^{|\alpha|}} \sum_{\beta \leqslant \alpha} \sum_{\gamma \leqslant \beta/2} \end{aligned}$$

$$\times |t|^{|\gamma|}\frac{\alpha!^{1-s}}{\beta!^{1-s}(\alpha-\beta)!^{1-s}}\frac{\beta!^{1-s}}{\gamma!(\beta-2\gamma)!}\frac{1}{(\alpha-\beta)!^s}\left\|\frac{x^{\beta-2\gamma}}{\langle x\rangle^{|\alpha|}}J^{\alpha-\beta}u\right\|_\theta$$

$$\leqslant \frac{2^{-|\alpha|/2}}{|\alpha|^{2\Theta}\rho_0^{|\alpha|}}\sum_{\beta\leqslant\alpha}\sum_{\gamma\leqslant\beta/2}$$

$$\times (1+T)^{|\beta|/2}\frac{\beta!^{1/2}}{\gamma!(\beta-2\gamma)!}\frac{1}{(\alpha-\beta)!^s}\left\|\frac{x^{\beta-2\gamma}}{\langle x\rangle^{|\alpha|}}J^{\alpha-\beta}u\right\|_\theta. \tag{50}$$

We remark that there exists a constant $C_0 > 0$ which is independent of α, β and γ, such that

$$\|x^{\beta-2\gamma}\langle x\rangle^{-|\alpha|}v\|_\theta \leqslant C_0|\alpha|^{2\Theta}\|v\|_\theta. \tag{51}$$

On the other hand, $\beta!^{1/2} \leqslant 2^{|\beta|/2}(\beta-[\beta/2])!$ since

$$\frac{\beta!}{(\beta-[\beta/2])!^2} \leqslant \frac{[\beta/2]!}{[\beta/2]!(\beta-[\beta])!} \leqslant 2^{|\beta|}.$$

Hence, we have

$$\begin{aligned}\sum_{\gamma\leqslant\beta/2}\frac{\beta!^{1/2}}{\gamma!(\beta-2\gamma)!} &\leqslant 2^{|\beta|/2}\sum_{\gamma\leqslant\beta/2}\frac{(\beta-[\beta/2])!}{\gamma!(\beta-2\gamma)!}\\ &\leqslant 2^{|\beta|/2}\sum_{\gamma\leqslant\beta/2}\frac{(\beta-[\beta/2])!}{\gamma!(\beta-[\beta/2]-\gamma)!}\\ &\leqslant 2^{|\beta|/2}\sum_{\gamma\leqslant\beta-[\beta/2]}\frac{(\beta-[\beta/2])!}{\gamma!(\beta-[\beta/2]-\gamma)!}\\ &\leqslant 2^{3|\beta|/2-[\beta/2]} \leqslant 2^{|\beta|+n}\end{aligned} \tag{52}$$

Using (47), (50), (51) and (52), we deduce

$$\begin{aligned}&\frac{1}{(|\alpha|+2\Theta)!^s}\left(\frac{|t|}{\rho_0}\right)^{|\alpha|}\|\langle x\rangle^{-|\alpha|}\partial^\alpha u\|_\theta\\ &\leqslant \frac{C_0 2^{-|\alpha|/2}}{\rho_0^{|\alpha|}}\sum_{\beta\leqslant\alpha}(1+T)^{|\beta|/2}\frac{1}{(\alpha-\beta)!^s}\|J^{\alpha-\beta}u\|_\theta\sum_{\gamma\leqslant\beta/2}\frac{\beta!^{1/2}}{\gamma!(\beta-2\gamma)!}\\ &\leqslant \frac{2^nC_0}{\rho_0^{|\alpha|}}\sum_{\beta\leqslant\alpha}\{2(1+T)\}^{|\beta|/2}\frac{1}{(\alpha-\beta)!^s}\|J^{\alpha-\beta}u\|_\theta\\ &\leqslant 2^nC_0\rho_0^{-|\alpha|/2}\sum_{\beta\leqslant\alpha}\left(\frac{2(1+T)}{\rho_0}\right)^{|\beta|/2}(\rho_0^{1/2}r)^{-|\alpha-\beta|}\frac{r^{|\alpha-\beta|}}{(\alpha-\beta)!^s}\|J^{\alpha-\beta}u\|_\theta\\ &\leqslant 2^nC_0\rho_0^{-|\alpha|/2}\sum_\beta\frac{r^{|\beta|}}{\beta!^s}\|J^\beta u\|_\theta\\ &\leqslant 2^nM_0\rho_0^{-|\alpha|/2} \leqslant 2^{n-|\alpha|}C_0M_0.\end{aligned}$$

Thus

$$\sum_\alpha\frac{1}{(|\alpha|+2\Theta)!^s}\left(\frac{|t|}{\rho_0}\right)^{|\alpha|}\|\langle x\rangle^{-|\alpha|}\partial^\alpha u\|_\theta \leqslant 2^{2n}C_0M_0.$$

If we set $\rho = 2^s\rho_0$, we have

$$\sum_\alpha\frac{1}{|\alpha|!^s}\left(\frac{|t|}{\rho}\right)^{|\alpha|}\|\langle x\rangle^{-|\alpha|}\partial^\alpha u\|_\theta$$

$$
\begin{aligned}
&\leqslant \frac{2^{2s\Theta}}{(2\Theta)!^s}\sum_{\alpha}\frac{1}{(|\alpha|+2\Theta)!^s}\left(\frac{2^s|t|}{\rho}\right)^{|\alpha|}\|\langle x\rangle^{-|\alpha|}\partial^\alpha u\|_\theta \\
&= \frac{2^{2s\Theta}}{(2\Theta)!^s}\sum_{\alpha}\frac{1}{(|\alpha|+2\Theta)!^s}\left(\frac{|t|}{\rho_0}\right)^{|\alpha|}\|\langle x\rangle^{-|\alpha|}\partial^\alpha u\|_\theta \\
&\leqslant \frac{2^{2n+2s\Theta}C_0a^nM}{(2\Theta)!^s}.
\end{aligned}
$$

This completes the proof. □

Proof of Lemma 17. When $|\alpha|+2m=0$, (48) and (49) are obvious. When $|\alpha|+2m\neq 0$, (48) follows from

$$
\begin{aligned}
\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^{\alpha+e_j}u = {}& \partial_j\Big(\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^\alpha u\Big) \\
& -(|\alpha|+2m)\frac{2x_j}{\langle x\rangle^2}\langle x\rangle^{-|\alpha|+2m}\partial_t^m\partial^\alpha u,
\end{aligned}
$$

and (49) follows from

$$
\begin{aligned}
\langle x\rangle^{-2}\partial_j\{\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^{\alpha+e_k}u\} = {}& \langle x\rangle^{-|\alpha|-2m-2}\partial_t^m\partial^{\alpha+e_j+e_k}u \\
& -(|\alpha|+2m)\frac{2x_j}{\langle x\rangle^3}\langle x\rangle^{-|\alpha|-2m-1}\partial_t^m\partial^{\alpha+e_k}u.
\end{aligned}
$$

This completes the proof. □

Now we shall complete the proof of Theorem 2.

Proof of Theorem 2. We shall obtain (5) from (1) and (47) by an induction argument on the order of the time derivatives. Suppose $s\geqslant 1$ Set

$$
Y^l(t)=\sum_{m=0}^{l}\sum_{\alpha}\frac{|t|^{|\alpha|+2m}\rho^{-|\alpha|}\kappa^{-2m}}{(|\alpha|+2m-2)_+!^s}\|\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^\alpha u(t)\|_\theta,
$$

with some $\kappa>0$ determined later. Lemma 16 shows that there exists a positive constant M such that $Y^0(t)\leqslant M/2$ for $t\in[-T,T]\setminus\{0\}$. Suppose that $Y^l(t)\leqslant M$ for $t\neq 0$. Since u is a solution to (1), we deduce

$$
\begin{aligned}
Y^{l+1}(t) &= Y^0(t)+\sum_{m=1}^{l+1}\sum_{\alpha}\frac{|t|^{|\alpha|+2m}\rho^{-|\alpha|}\kappa^{-2m}}{(|\alpha|+2m-2)!^s}\|\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^\alpha u(t)\|_\theta \\
&= Y^0(t) \\
&\quad+\sum_{m=0}^{l}\sum_{\alpha}\frac{|t|^{|\alpha|+2m+2}\rho^{-|\alpha|}\kappa^{-2m-2}}{(|\alpha|+2m)!^s}\|\langle x\rangle^{-|\alpha|-2m-2}\partial_t^m\partial^\alpha\partial_t u(t)\|_\theta \\
&\leqslant Y^0(t) \\
&\quad+\sum_{j=1}^{n}\sum_{m=0}^{l}\sum_{\alpha}\frac{|t|^{|\alpha|+2m+2}\rho^{-|\alpha|}\kappa^{-2m-2}}{(|\alpha|+2m)!^s}\|\langle x\rangle^{-|\alpha|-2m-2}\partial_t^m\partial^{\alpha+2e_j}u(t)\|_\theta \\
&\quad+\sum_{m=0}^{l}\sum_{\alpha}\frac{|t|^{|\alpha|+2m+2}\rho^{-|\alpha|}\kappa^{-2m-2}}{(|\alpha|+2m)!^s}\|\langle x\rangle^{-|\alpha|-2m-2}\partial_t^m\partial^\alpha f(u,\partial u)(t)\|_\theta \\
&\leqslant Y^0(t)+\frac{\rho^2 n}{\kappa^2}Y^l(t)+Z^l(t)
\end{aligned}
$$

$$\leqslant \left(\frac{1}{2}+\frac{\rho^2 n}{\kappa^2}\right) M + Z^l(t) \tag{53}$$

where

$$Z^l(t) = \sum_{m=0}^{l}\sum_{\alpha} \frac{|t|^{|\alpha|+2m+2}\rho^{-|\alpha|}\kappa^{-2m-2}}{(|\alpha|+2m)!^s} \|\langle x\rangle^{-|\alpha|-2m-2}\partial_t^m \partial^\alpha f(u,\partial u)(t)\|_\theta .$$

The chain rule shows that

$$\begin{aligned}\partial_t^m \partial^\alpha f(u,\partial u) = & \sum_{p=1}^{\infty}\sum_{\substack{|\gamma|=p+1\\|\bar\gamma|=p}} f_{\gamma\bar\gamma} \sum_{\substack{m_0+\cdots+m_{2p}=m\\ \alpha^0+\cdots+\alpha^{2p}=\alpha}} \\ & \times \frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{m!}{m_0!\cdots m_{2p}!}\prod_{q=0}^{2p}\partial_t^{m_q}\partial^{\alpha^q}\partial_{q,\gamma\bar\gamma}\tilde u,\end{aligned}$$

where $\tilde u = u$ or $\bar u$. Set $\partial_0 = 1$ for short. Using this and $[\partial_j, \langle x\rangle^{-2}] = \mathcal{O}(\langle x\rangle^{-2})$ ($j = 0, 1, \dots, n$), we have

$$\begin{aligned} Z^l(t) \leqslant & \sum_{m=0}^{l}\sum_{\alpha} \frac{|t|^{|\alpha|+2m+2}\rho^{-|\alpha|}\kappa^{-2m-2}}{(|\alpha|+2m)!^s} \\ & \times \sum_{p=1}^{\infty}\sum_{\substack{|\gamma|=p+1\\|\bar\gamma|=p}} |f_{\gamma\bar\gamma}| \sum_{\substack{m_0+\cdots+m_{2p}=m\\ \alpha^0+\cdots+\alpha^{2p}=\alpha}} \frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{m!}{m_0!\cdots m_{2p}!} \\ & \times \sum_{j=0}^{n}\left\|\langle x\rangle^{-2}\partial_j\left\{\prod_{q=0}^{2p}\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}\partial_{q,\gamma\bar\gamma}\tilde u\right\}\right\|_{\theta-1}. \end{aligned} \tag{54}$$

By using Lemmas 7 and 17, we deduce

$$\begin{aligned} & \left\|\langle x\rangle^{-2}\partial_0\left\{\prod_{q=0}^{2p}\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}\partial_{q,\gamma\bar\gamma}\tilde u\right\}\right\|_{\theta-1} \\ & \leqslant \left\|\prod_{q=0}^{2p}\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}\partial_{q,\gamma\bar\gamma}\tilde u\right\|_{\theta-1} \\ & \leqslant (2p+1)C_0^{2p+1}\prod_{q=0}^{2p}\|\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}\partial_{q,\gamma\bar\gamma}\tilde u\|_{\theta-1} \\ & \leqslant (2p+1)C_0^{4p+2}\prod_{q=0}^{2p}(|\alpha^q|+2m_q)_+\|\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}u\|_\theta \qquad (55) \\ & = (2p+1)C_0^{4p+2}|t|^{-|\alpha|-2m}\rho^{|\alpha|}\kappa^{2m}\prod_{q=0}^{2p}(|\alpha^q|+2m_q)!^s \\ & \quad \times \prod_{q=0}^{2p}\frac{|t|^{|\alpha^q|+2m_q}\rho^{-|\alpha^q|}\kappa^{-2m_q}}{(|\alpha^q|+2m_q-2)_+!^s}\|\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}u\|_\theta, \qquad (56) \end{aligned}$$

and for $j \neq 0$

$$
\begin{aligned}
&\left\| \langle x \rangle^{-2} \partial_j \left\{ \prod_{q=0}^{2p} \langle x \rangle^{-|\alpha^q|-2m_q} \partial_t^{m_q} \partial^{\alpha^q} \partial_{q,\gamma\bar{\gamma}} \tilde{u} \right\} \right\|_{\theta-1} \\
&\leqslant \sum_{r=0}^{2p} \Bigg\| \prod_{\substack{q=0 \\ q\neq r}}^{2p} \langle x \rangle^{-|\alpha^q|-2m_q} \partial_t^{m_q} \partial^{\alpha^q} \partial_{q,\gamma\bar{\gamma}} \tilde{u} \\
&\qquad \times \langle x \rangle^{-2} \partial_j \{ \langle x \rangle^{-|\alpha^r|-2m_r} \partial_t^{m_r} \partial^{\alpha^r} \partial_{r,\gamma\bar{\gamma}} \tilde{u} \} \Bigg\|_{\theta-1} \\
&\leqslant (2p+1) C_0^{2p+1} \sum_{r=0}^{2p} \prod_{\substack{q=0 \\ q\neq r}}^{2p} \| \langle x \rangle^{-|\alpha^q|-2m_q} \partial_t^{m_q} \partial^{\alpha^q} \partial_{q,\gamma\bar{\gamma}} \tilde{u} \|_{\theta-1} \\
&\qquad \times \| \langle x \rangle^{-2} \partial_j \{ \langle x \rangle^{-|\alpha^r|-2m_r} \partial_t^{m_r} \partial^{\alpha^r} \partial_{r,\gamma\bar{\gamma}} \tilde{u} \} \|_{\theta-1} \\
&\leqslant (2p+1) C_0^{4p+2} \sum_{r=0}^{2p} \prod_{\substack{q=0 \\ q\neq r}}^{2p} (|\alpha^q| + 2m_q)_+ \| \langle x \rangle^{-|\alpha^q|-2m_q} \partial_t^{m_q} \partial^{\alpha^q} u \|_\theta \\
&\qquad \times \Big\{ \| \langle x \rangle^{-|\alpha^r+e_j+e'_r|-2m_r} \partial_t^{m_r} \partial^{\alpha^r+e_j+e'_r} u \|_{\theta-1} \\
&\qquad\qquad + (|\alpha^r| + 2m_r)_+ \| \langle x \rangle^{-|\alpha^r+e'_r|-2m_r} \partial_t^{m_r} \partial^{\alpha^r+e'_r} u \|_{\theta-1} \Big\} \\
&= (2p+1) C_0^{4p+2} |t|^{-|\alpha|-2m-1} \rho^{|\alpha|+1} \kappa^{2m} \prod_{q=0}^{2p} (|\alpha^q| + 2m_q)!^s \sum_{r=0}^{2p} \\
&\qquad \times \prod_{\substack{q=0 \\ q\neq r}}^{2p} \frac{|t|^{|\alpha^q|} \rho^{-|\alpha^q|} \kappa^{-2m_q}}{(|\alpha^q| + 2m_q - 2)_+!^s} \| \langle x \rangle^{-|\alpha^q|-2m_q} \partial_t^{m_q} \partial^{\alpha^q} u \|_\theta \\
&\qquad\quad \frac{|t|^{|\alpha^r+e_j|} \rho^{-|\alpha^r+e_j|} \kappa^{-2m_r}}{(|\alpha^r + e_j| + 2m_r - 2)_+!^s} \| \langle x \rangle^{-|\alpha^r+e_j|-2m_r} \partial_t^{m_r} \partial^{\alpha^r+e^j} u \|_\theta \\
&+ (2p+1) C_0^{4p+2} |t|^{-|\alpha|-2m} \rho^{|\alpha|} \kappa^{2m} \prod_{q=0}^{2p} (|\alpha^q| + 2m_q)!^s \\
&\qquad \times \prod_{q=0}^{2p} \frac{|t|^{|\alpha^q|+2m_q} \rho^{-|\alpha^q|} \kappa^{-2m_q}}{(|\alpha^q| + 2m_q - 2)_+!^s} \| \langle x \rangle^{-|\alpha^q|-2m_q} \partial_t^{m_q} \partial^{\alpha^q} u \|_\theta,
\end{aligned}
\tag{57}
$$

where $e'_r = e_j$ with some $j = 0, 1, \dots, n$. Substituting (56) and (57) into (54), we have

$$
\begin{aligned}
Z^l(t) \leqslant \frac{CT^2}{\kappa^2} \sum_{p=1}^{\infty} \sum_{\substack{|\gamma|=p+1 \\ |\bar{\gamma}|=p}} |f_{\gamma\bar{\gamma}}| (1 + C_0^2)^{2p+1} (2p+1)^2 \\
\times \sum_{m=0}^{l} \sum_{\alpha} \sum_{\substack{m_0+\cdots+m_{2p}=m \\ \alpha^0+\cdots+\alpha^{2p}=\alpha}} \frac{\prod_{q=0}^{2p} (|\alpha^q| + 2m_q)!^s}{(|\alpha| + 2m)!^s} \frac{\alpha!}{\alpha^0! \cdots \alpha^{2p}!} \frac{m!}{m_0! \cdots m_{2p}!}
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{q=0}^{2p} \frac{|t|^{|\alpha^q|+2m_q}\rho^{-|\alpha^q|}\kappa^{-2m_q}}{(|\alpha^q|+2m_q-2)_+!^s} \|\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}u\|_\theta \\
&+ \sum_{j=1}^n \frac{CT\rho}{\kappa^2}\sum_{p=1}^\infty \sum_{\substack{|\gamma|=p+1\\|\bar{\gamma}|=p}} |f_{\gamma\bar{\gamma}}|(1+C_0^2)^{2p+1}(2p+1)\sum_{r=0}^{2p} \\
&\times \sum_{m=0}^l \sum_\alpha \sum_{\substack{m_0+\cdots+m_{2p}=m\\ \alpha^0+\cdots+\alpha^{2p}=\alpha}} \frac{\displaystyle\prod_{q=0}^{2p}(|\alpha^q|+2m_q)!^s}{(|\alpha|+2m)!^s}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{m!}{m_0!\cdots m_{2p}!} \\
&\times \prod_{\substack{q=0\\q\neq r}}^{2p} \frac{|t|^{|\alpha^q|+2m_q}\rho^{-|\alpha^q|}\kappa^{-2m_q}}{(|\alpha^q|+2m_q-2)_+!^s} \|\langle x\rangle^{-|\alpha^q|-2m_q}\partial_t^{m_q}\partial^{\alpha^q}u\|_\theta \\
&\times \frac{|t|^{|\alpha^r+e_j|+2m_r}\rho^{-|\alpha^r+e_j|}\kappa^{-2m_r}}{(|\alpha^r|+2m_r-2)_+!^s}\|\langle x\rangle^{-|\alpha^r+e_j|-2m_r}\partial_t^{m_r}\partial^{\alpha^r+e_j}u\|_\theta. \qquad (58)
\end{aligned}
$$

In view of Lemma 9, we have

$$
\begin{aligned}
&\frac{\displaystyle\prod_{q=0}^{2p}(|\alpha^q|+2m_q)!^s}{(|\alpha|+2m)!^s}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{m!}{m_0!\cdots m_{2p}!} \\
&\leqslant \frac{\displaystyle\prod_{q=0}^{2p}(|\alpha^q|+2m_q)!^s}{(|\alpha|+2m)!^s}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{m!}{m_0!\cdots m_{2p}!} \\
&\leqslant \frac{\displaystyle\prod_{q=0}^{2p}(|\alpha^q|+2m_q)!^s}{(|\alpha|+2m)!^s}\frac{\alpha!}{\alpha^0!\cdots\alpha^{2p}!}\frac{(2m)!}{(2m_0)!\cdots(2m_{2p})!} \\
&= \frac{\displaystyle\prod_{q=0}^{2p}|(\alpha^q,2m_q)|!^s}{|(\alpha,2m)|!^s}\frac{(\alpha,2m)!}{\displaystyle\prod_{q=0}^{2p}(\alpha^q,2m_q)!} \leqslant 1.
\end{aligned}
$$

Applying this to (58), we deduce

$$
\begin{aligned}
Z^l(t) &\leqslant \frac{CT(T+\rho)}{\kappa^2}\sum_{p=1}^\infty A_p(1+C_0^2)^{2p+1}(2p+1)^2 Y^l(t)^{2p+1} \\
&\leqslant \frac{2CT(T+\rho)}{\kappa^2}\sum_{p=1}^\infty A_p(1+C_0^2)^{2p+1}(2p+1)^2\Big\{e(1+C_0)^2M\Big\}^{2p+1}.
\end{aligned}
$$

Set $E=C_R$ with $R=2e(1+C_0)^2M$ for short. Then we have

$$
Z^l(t) \leqslant \frac{2CT(T+\rho)E}{\kappa^2} \qquad (59)
$$

for $t \in [-T,T] \setminus \{0\}$. Combining (53) and (59), we have

$$Y^{l+1}(t) \leqslant \frac{M}{2} + \frac{\rho^2 nM + 2CT(T+\rho)E}{\kappa^2}$$

for $t \in [-T,T] \setminus \{0\}$. If we choose κ satisfying

$$\kappa \geqslant \sqrt{\frac{2\rho^2 nM + 4CT(T+\rho)E}{M}},$$

then $Y^{l+1}(t) \leqslant M$ for $t \in [-T,T] \setminus \{0\}$. This completes the proof. □

8. Concluding remarks

Finally we state some remarks concerned with the results of [7]. We shall present some examples for which the Gevrey estimate (5) holds for $s \geqslant 1/2$. First we remark that $e^{it\Delta}u_0$ gains analyticity in space-time variables if u_0 decays faster than the Gaussian functions.

Theorem 18. *Let $s \geqslant 1/2$ and $\theta \in \mathbb{R}$. Suppose that $\exp(\varepsilon\langle x\rangle^{1/s})u_0 \in H^\theta$ with some $\varepsilon > 0$. Then, For any $T > 0$ there exist positive constants M and ρ such that for $t \in [-T,T] \setminus \{0\}$*

$$\|\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^\alpha e^{it\Delta}u_0\|_\theta \leqslant M\rho^{|\alpha|+2m}t^{-|\alpha|-2m}m!^{2s}\alpha!^s.$$

Proof. Fix arbitrary $T > 0$. Lemma 8 shows that

$$\|x^\alpha u_0\|_\theta \leqslant M_0\rho_0^{|\alpha|}\alpha!^s$$

with some $M_0 > 0$ and $\rho_0 > 0$. Applying J^α to $(\partial_t - i\Delta)e^{it\Delta}u_0 = 0$, we have $(\partial_t - i\Delta)J^\alpha u = 0$. It is easy to see that

$$\|J^\alpha e^{it\Delta}u_0\|_\theta = \|x^\alpha u_0\|_\theta \leqslant M_0\rho_0^{|\alpha|}\alpha!^s.$$

Lemma 16 shows that for $t \in [-T,T] \setminus \{0\}$

$$\|\langle x\rangle^{-|\alpha|}\partial^\alpha e^{it\Delta}u_0\|_\theta \leqslant M_1\rho_1^{|\alpha|}|t|^{-|\alpha|}\alpha!^s$$

with some $M_1 > 0$ and $\rho_1 > 0$. Using the equation $(\partial_t - i\Delta)e^{it\Delta}u_0 = 0$ again, we deduce

$$\begin{aligned}
&\|\langle x\rangle^{-|\alpha|-2m}\partial_t^m\partial^\alpha e^{it\Delta}u_0\|_\theta \\
&= \|\langle x\rangle^{-|\alpha|-2m}\Delta^m\partial^\alpha e^{it\Delta}u_0\|_\theta \\
&\leqslant \sum_{j(1)=1}^{n}\cdots\sum_{j(m)=1}^{n}\|\langle x\rangle^{-|\alpha|-2m}\partial^{\alpha+2(e_{j(1)}+\cdots+e_{j(m)})}e^{it\Delta}u_0\|_\theta \\
&\leqslant n^m M_1\rho_1^{|\alpha|+2m}t^{-|\alpha|-2m}(|\alpha|+2m)!^s
\end{aligned}$$

for $t \in [-T,T] \setminus \{0\}$. This completes the proof. □

Next we apply Theorem 18 to the initial value problem for one-dimensional nonlinear equations of the form

$$u_t - iu_{xx} = 2a(|u|^2)_x u + ia^2|u|^4 u \quad \text{in} \quad \mathbb{R}^2, \tag{60}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}, \tag{61}$$

where $u_t = \partial u/\partial t$, $u_{xx} = \partial^2 u\partial/x^2$, and a is a real constant. The equation (60) has very special nonlinearity. In fact, if u is a smooth solutions to (60), then

$$v(t,x) = \exp\left(-ia\int_{-\infty}^{x}|u(t,y)|^2 dy\right)u(t,x) \tag{62}$$

formally solves the equation $v_t - iv_{xx} = 0$. The mapping

$$u \longmapsto v(x) = \exp\left(-ia\int_{-\infty}^{x}|u(y)|^2dy\right)u(x) \tag{63}$$

defined for functions of x is said to be a gauge transform. We remark that $|u(x)| = |v(x)|$ and $\|u\| = \|v\|$ for $u \in L^2(\mathbb{R})$, and the inverse of the gauge transform is given by

$$u(x) = \exp\left(ia\int_{-\infty}^{x}|v(y)|^2dy\right)v(x). \tag{64}$$

More properties of the gauge transform needed in this section are the following.

Lemma 19. *Let $\theta > 1/2$ and $s > 0$.*

(i) **:** *The gauge transform is a homeomorphic mapping of $H^\theta(\mathbb{R})$ onto itself.*

(ii) **:** *If $u \in H^\theta(\mathbb{R})$ satisfies $\exp(\varepsilon\langle x\rangle^{1/s})u \in H^\theta(\mathbb{R})$, the gauge transformation of u has the same property.*

(iii) **:** *If $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ solves (60), then $v(t,x)$ defined by (62) solves $v_t - iv_{xx} = 0$.*

(iv) **:** *If the sequence $\{u_n\}_{n=1}^\infty \subset H^\theta(\mathbb{R})$ satisfies*

$$u_n \longrightarrow u \quad \textit{in} \quad H^\theta(\mathbb{R}) \quad \textit{as} \quad n \to \infty,$$

then

$$(|u_n|^2)_x u_n \longrightarrow (|u|^2)_x u \quad \textit{in} \quad \mathscr{D}'(\mathbb{R}) \quad \textit{as} \quad n \to \infty,$$

where $\mathscr{D}'(\mathbb{R})$ is the space of distributions on $\mathbb{R}$.

Proof. Set

$$\phi(x) = \int_{-\infty}^{x}|u(y)|^2dy$$

for short. First we show (i). Suppose that $u \in H^\theta(\mathbb{R})$ with some $\theta > 1/2$. We can check $\phi \in \mathscr{B}^{\theta+1/2}(\mathbb{R})$ and $\phi' \in H^\theta$ since $\phi'(x) = |u(x)|^2$ and $H^\theta(\mathbb{R})$ is an algebra for $\theta > 1/2$. For any integer $j = 0, 1, 2, \cdots, [\theta]$, the chain rule shows that

$$\begin{aligned}\left(\frac{d}{dx}\right)^j(e^{-ia\phi}u) &= \sum_{k=0}^{j}\frac{j!}{k!(j-k)!}\left(\frac{d}{dx}\right)^k e^{-ia\phi}\left(\frac{d}{dx}\right)^{j-k}u \\ &= \sum_{k=0}^{j}\frac{j!}{k!(j-k)!}e^{-ia\phi}\left\{e^{ia\phi}\left(\frac{d}{dx}\right)^k e^{-ia\phi}\right\}\left(\frac{d}{dx}\right)^{j-k}u. \end{aligned} \tag{65}$$

In view of (9), we deduce

$$e^{-ia\phi}\left\{e^{ia\phi}\left(\frac{d}{dx}\right)^k e^{-ia\phi}\right\}\left(\frac{d}{dx}\right)^{j-k}u \in H^{\theta-j}(\mathbb{R}). \tag{66}$$

Lemma 4 shows that

$$\langle D\rangle^{\theta-[\theta]}\left(\frac{d}{dx}\right)^j(e^{-ia\phi}u) \in H^{[\theta]-j}(\mathbb{R}) \quad \text{for} \quad j = 0, 1, \dots, [\theta].$$

This asserts that $u \in H^\theta(\mathbb{R}) \mapsto e^{-ia\phi}u \in H^\theta(\mathbb{R})$ is continuous. In the same way, the inverse (64) is also continuous. Hence the gauge transform (63) is homeomorphic on $H^\theta(\mathbb{R})$.

Next we show (ii). Replacing u by $\exp(\varepsilon\langle x\rangle^{1/s})u$ in (65), we have

$$\left(\frac{d}{dx}\right)^j\left(\exp(\varepsilon\langle x\rangle^{1/s})e^{-ia\phi}u\right)$$

$$= \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} e^{-ia\phi} \left\{ e^{ia\phi} \left(\frac{d}{dx} \right)^k e^{-ia\phi} \right\} \left(\frac{d}{dx} \right)^{j-k} \exp(\varepsilon \langle x \rangle^{1/s}) u.$$

In the same way, we can check $\exp(\varepsilon\langle x\rangle^{1/s})e^{-ia\phi}u \in H^\theta(\mathbb{R})$.

Next we show (iii). Suppose that $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ solves (60). It follows that $\partial_t u \in C(\mathbb{R}; H^{-1}(\mathbb{R}))$, $\phi \in C(\mathbb{R}; \mathscr{B}^{3/2}(\mathbb{R}))$, $\phi_x \in C(\mathbb{R}; H^1(\mathbb{R}))$ and $v = e^{-ia\phi}u$ also belongs to $C(\mathbb{R}; H^1(\mathbb{R}))$. Thus, the following computations

$$\begin{aligned}
e^{-ia\phi} u_t &= v_t + iav\phi_t \\
&= v_t + iav \int_{-\infty}^{x} (u_t \bar{u} + u \bar{u}_t + 4a(|u|^2)_x |u|^2 + ia^2|u|^6 - ia^2|u|^r) dy \\
&= v_t + iav \int_{-\infty}^{x} (iu_{xx}\bar{u} - iu\bar{u}_{xx} + 2a(|u|^4)_x) dy \\
&= v_t - a(u_x\bar{u} - u\bar{u}_x)v + 2ia^2|u|^4 v, \\
v_x &= -ia|u|^2 v + e^{-ia\phi}u_x, \\
-ie^{-ia\phi}u_{xx} &= -iv_{xx} + 2a|u|^2 v_x + a(|u|^2)_x + ia^2|u|^4 v \\
&= -iv_{xx} + a(|u|^2)_x + 2au_x\bar{u}v - ia|u|^4 v,
\end{aligned}$$

are justified, and it is easy to see that v solves $v_t - iv_{xx} = 0$.

Lastly, we check (iv). Fix arbitrary $\psi \in \mathscr{D}(\mathbb{R})$. Since $\theta > 1/2$, one can easily verify

$$(|u_n|^2)_x \to (|u|^2)_x \quad \text{in} \quad H^{-1/2}(\mathbb{R}), \qquad u_n\psi \to u\psi \quad \text{in} \quad H^{1/2}(\mathbb{R}).$$

This shows

$$(|u_n|^2)_x u_n \longrightarrow (|u|^2)_x u \quad \text{in} \quad \mathscr{D}'(\mathbb{R}) \quad \text{as} \quad n \to \infty.$$

This completes the proof. □

Theorem 18 and Lemma 19 prove the following.

Theorem 20. *Let $\theta \geqslant 1$, $s \geqslant 1/2$ and $\varepsilon > 0$. Set $\sigma = \max\{1, s\}$.*

Existence **:** *Suppose that $u_0 \in H^\theta(\mathbb{R})$. Then, the initial value problem* (60)-(61) *posseses a unique solution $u \in C(\mathbb{R}; H^\theta(\mathbb{R}))$.*

Analyticity **:** *Moreover, if $\exp(\varepsilon\langle x\rangle^{1/s})u_0 \in H^\theta(\mathbb{R})$, then for any $T > 0$ there exist positive constants M and ρ such that for $t \in [-T, T] \setminus \{0\}$*

$$\|\langle x\rangle^{-\alpha-2m}\partial_t^m \partial^\alpha u(t)\|_\theta \leqslant M\rho^{\alpha+2m}|t|^{-\alpha-2m} m!^{2s} \alpha!^\sigma.$$

Proof of Theorem 20, *Existence.* Set $v = e^{it\partial^2}(e^{-ia\phi_0}u_0)$, $u = e^{ia\phi}v$,

$$\phi_0(x) = \int_{-\infty}^{x} |u_0(y)|^2 dy, \quad \phi(t,x) = \int_{-\infty}^{x} |v(t,y)|^2 dy.$$

Here we remark that $|u(t,x)| = |v(t,x)|$, $\|u(t)\| = \|v(t)\|$. Lemma 19 shows that $v \mapsto u$ is homeomorphic on $C(\mathbb{R}; H^\theta(\mathbb{R}))$. Pick up a sequence $\{v_0^{(n)}\}_{n=1}^\infty \subset \mathscr{S}(\mathbb{R})$ satisfying

$$\|v_0^{(n)} - e^{-ia\phi_0}u_0\|_\theta \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Set $v^{(n)} = e^{iy\partial^2}v_0^{(n)}$, $u^{(n)} = e^{ia\phi^{(n)}}v^{(n)}$,

$$\phi^{(n)}(t,x) = \int_{-\infty}^{x} |v^{(n)}(t,y)|^2 dy, \quad \phi_0^{(n)}(x) = \int_{-\infty}^{x} |v_0^{(n)}(y)|^2 dy.$$

Since $v^{(n)} \in C^\infty(\mathbb{R}; \mathscr{S}(\mathbb{R}))$, $u^{(n)}$ solves

$$u_t^{(n)} - iu_{xx}^{(n)} = 2a(|u^{(n)}|^2)_x u^{(n)} + ia^2|u^{(n)}|^4 u^{(n)},$$

$$u^{(n)}(0,x)=e^{ia\phi_0^{(n)}(x)}v_0^{(n)}(x).$$

Obviously,

$$\|v^{(n)}(t)-v(t)\|_\theta=\|v_0^{(n)}-e^{-ia\phi_0}u_0\|_\theta\longrightarrow 0\quad\text{as}\quad n\to\infty$$

for any $t\in\mathbb{R}$. Using Lemma 19 again, we deduce that

$$\|u^{(n)}(t)-u(t)\|_\theta\longrightarrow 0\quad\text{as}\quad n\to\infty$$

for any $t\in\mathbb{R}$, and that u is a solution to (60)-(61). The uniqueness of v implies the uniqueness of u. □

Proof of Theorem 20, *Analyticity.* Fix arbitrary $T>0$. We remark that the solution u is represented by $u=e^{ia\phi}e^{it\partial^2}(e^{-ia\phi_0}u_0)$. By using Theorem 18 and Lemma 19, we heve

$$\|\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha v(t)\|_\theta\leqslant M\rho^{\alpha+2m}|t|^{-\alpha-2m}(\alpha+2m)!^s \tag{67}$$

for $t\in[-T,T]\setminus\{0\}$. Note that

$$\|u_0\|=\|v_0\|\leqslant\|v_0\|_\theta=\|v(t)\|_\theta\leqslant M.$$

First we shall show that for $\alpha+2m\geqslant 1$ and $t\in[-T,T]\setminus\{0\}$

$$\|\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha\phi(t)\|_\theta\leqslant C_\theta M^2(2\rho)^{\alpha+2m-1}|t|^{-\alpha-2m+1}(\alpha+2m)!^s. \tag{68}$$

When $m=0$ and $\alpha\geqslant 1$,

$$\partial^\alpha\phi=\partial^{\alpha-1}|v|^2=\sum_{\beta=0}^{\alpha-1}\frac{(\alpha-1)!}{\beta!(\alpha-\beta-1)!}\partial^\beta v\partial^{\alpha-\beta-1}\bar{v}.$$

By using this formula and the fact that $H^\theta(\mathbb{R})$ is an algebra, we have

$$\begin{aligned}\|\langle x\rangle^{-\alpha}\partial^\alpha\phi(t)\|_\theta &\leqslant\sum_{\beta=0}^{\alpha-1}\frac{(\alpha-1)!}{\beta!(\alpha-\beta-1)!}\|\langle x\rangle^{-\alpha}\partial^\beta v\partial^{\alpha-\beta-1}\bar{v}\|_\theta\\ &\leqslant C_\theta\sum_{\beta=0}^{\alpha-1}\frac{(\alpha-1)!}{\beta!(\alpha-\beta-1)!}\|\langle x\rangle^{-\alpha+1}\partial^\beta v\partial^{\alpha-\beta-1}\bar{v}\|_\theta\\ &\leqslant C_\theta\sum_{\beta=0}^{\alpha-1}\frac{(\alpha-1)!}{\beta!(\alpha-\beta-1)!}\|\langle x\rangle^{-\beta}\partial^\beta v\|_\theta\|\langle x\rangle^{-\alpha+\beta+1}\partial^{\alpha-\beta-1}\bar{v}\|_\theta\\ &\leqslant C_\theta M^2\rho^{\alpha-1}|t|^{-\alpha+1}\sum_{\beta=0}^{\alpha-1}\frac{(\alpha-1)!}{\beta!(\alpha-\beta-1)!}\beta!^s(\alpha-\beta-1)!^s\\ &\leqslant C_\theta M^2\rho^{\alpha-1}|t|^{-\alpha+1}\alpha!^s\sum_{\beta=0}^{\alpha-1}\frac{(\alpha-1)!}{\beta!(\alpha-\beta-1)!}\\ &=C_\theta M^2(2\rho)^{\alpha-1}|t|^{-\alpha+1}\alpha!^s\end{aligned} \tag{69}$$

for $\alpha\geqslant 1$ and $t\in\mathbb{R}\setminus\{0\}$. On the other hand, when $m\geqslant 1$,

$$\begin{aligned}\partial_t^m\partial^\alpha\phi&=\partial_t^{m-1}\partial^\alpha\int_{-\infty}^x\partial_t|v|^2dy\\ &=i\partial_t^{m-1}\partial^\alpha\int_{-\infty}^x(v_{yy}\bar{v}-v\bar{v}_{yy})dy\\ &=i\partial_t^{m-1}\partial^\alpha(v_x\bar{v}-v\bar{v}_x)\\ &=2\,\mathrm{Im}\sum_{l=0}^{m-1}\sum_{\beta=0}^{\alpha}\frac{(m-1)!}{l!(m-l-1)!}\frac{\alpha!}{\beta!(\alpha-\beta)!}\partial_t^l\partial^{\beta+1}v\partial_t^{m-l-1}\partial^{\alpha-\beta}\bar{v}.\end{aligned}$$

Then, we have

$$\begin{aligned}
&\|\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha\phi(t)\|_\theta \\
&\leqslant 2\sum_{l=0}^{m-1}\sum_{\beta=0}^{\alpha}\frac{(m-1)!}{l!(m-l-1)!}\frac{\alpha!}{\beta!(\alpha-\beta)!}\|\langle x\rangle^{-\alpha-2m}\partial_t^l\partial^{\beta+1}v\partial_t^{m-l-1}\partial^{\alpha-\beta}\bar{v}\|_\theta \\
&\leqslant 2C_\theta\sum_{l=0}^{m-1}\sum_{\beta=0}^{\alpha}\frac{(m-1)!}{l!(m-l-1)!}\frac{\alpha!}{\beta!(\alpha-\beta)!}\|\langle x\rangle^{-\alpha-2m+1}\partial_t^l\partial^{\beta+1}v\partial_t^{m-l-1}\partial^{\alpha-\beta}\bar{v}\|_\theta \\
&\leqslant 2C_\theta\sum_{l=0}^{m-1}\sum_{\beta=0}^{\alpha}\frac{(m-1)!}{l!(m-l-1)!}\frac{\alpha!}{\beta!(\alpha-\beta)!} \\
&\qquad\times\|\langle x\rangle^{-\beta-2l-1}\partial_t^l\partial^{\beta+1}v\|_\theta\|\langle x\rangle^{-\alpha+\beta-2m+2l+2}\partial_t^{m-l-1}\partial^{\alpha-\beta}\bar{v}\|_\theta \\
&\leqslant 2C_\theta M^2\rho^{\alpha+2m-1}|t|^{-\alpha-2m+1}\sum_{l=0}^{m-1}\sum_{\beta=0}^{\alpha}\frac{(m-1)!}{l!(m-l-1)!}\frac{\alpha!}{\beta!(\alpha-\beta)!} \\
&\qquad\times(\beta+1+2l)!^s(\alpha-\beta-2m+2l-2)!^s \\
&\leqslant 2C_\theta M^2\rho^{\alpha+2m-1}|t|^{-\alpha-2m+1}(\alpha+2m-1)!^s\sum_{l=0}^{m-1}\sum_{\beta=0}^{\alpha}\frac{(m-1)!}{l!(m-l-1)!}\frac{\alpha!}{\beta!(\alpha-\beta)!} \\
&= 2^{\alpha+m}C_\theta M^2\rho^{\alpha+2m-1}|t|^{-\alpha-2m+1}(\alpha+2m-1)!^s \\
&\leqslant C_\theta M^2(2\rho)^{\alpha+2m-1}|t|^{-\alpha-2m+1}(\alpha+2m-1)!^s
\end{aligned} \tag{70}$$

for $m\geqslant 1$ and $t\in[-T,T]\setminus\{0\}$. Combining (69) and (70), we obtain (68).

Set

$$A = M+\left\{1+\frac{T}{2\rho}\right\}C_\theta M^2.$$

If we replace 2ρ by ρ, we have

$$\|\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha\phi(t)\|_\theta \leqslant A\rho^{\alpha+2m}|t|^{-\alpha-2m}m!^{2s}\alpha!^\sigma, \tag{71}$$

$$\|\phi(t)\|_{\mathscr{B}^{\theta+1/2}} \leqslant A, \tag{72}$$

for $t\in[-T,T]\setminus\{0\}$ and $\alpha+2m\neq 0$. We compute the regularity of u. The Taylor series of the exponential function gives

$$\begin{aligned}
\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha u &= \langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha(e^{ia\phi}v) \\
&= \sum_{k=0}^{\infty}\frac{(ia)^k}{k!}\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha\left(\phi^k u\right) \\
&= \sum_{k=0}^{\infty}\frac{(ia)^k}{k!}\sum_{\substack{m_0+\cdots+m_k=m\\ \alpha_0+\cdots+\alpha_k=\alpha}}\frac{m!}{m_0!\cdots m_k!}\frac{\alpha!}{\alpha_0!\cdots\alpha_k!} \\
&\qquad\times\langle x\rangle^{-\alpha_0-2m_0}\partial_t^{m_0}\partial^{\alpha_0}v\prod_{j=1}^{k}\langle x\rangle^{-\alpha_j-2m_j}\partial_t^{m_j}\partial^{\alpha_j}\phi.
\end{aligned}$$

Applying Lemma 18 to v, (71) to ϕ for $\alpha_j+2m_j\neq 0$, and (72) to ϕ fot $\alpha_j+2m_j=0$ respectively, we deduce

$$\|\langle x\rangle^{-\alpha-2m}\partial_t^m\partial^\alpha u(t)\|_\theta$$

$$
\begin{aligned}
&\leqslant \sum_{k=0}^{\infty} \frac{|a|^k}{k!} \sum_{\substack{m_0+\cdots+m_k=m \\ \alpha_0+\cdots+\alpha_k=\alpha}} \frac{m!}{m_0!\cdots m_k!} \frac{\alpha!}{\alpha_0!\cdots\alpha_k!} \\
&\qquad \times \left\| \langle x\rangle^{-\alpha_0-2m_0} \partial_t^{m_0} \partial^{\alpha_0} v(t) \prod_{j=1}^{k} \langle x\rangle^{-\alpha_j-2m_j} \partial_t^{m_j} \partial^{\alpha_j} \phi(t) \right\|_{\theta} \\
&\leqslant \sum_{k=0}^{\infty} \frac{|a|^k}{k!} \sum_{\substack{m_0+\cdots+m_k=m \\ \alpha_0+\cdots+\alpha_k=\alpha}} \frac{m!}{m_0!\cdots m_k!} \frac{\alpha!}{\alpha_0!\cdots\alpha_k!} \\
&\qquad \times C_\theta^{k+1} A^{k+1} \rho^{\alpha+2m} |t|^{-\alpha+2m} \prod_{j=0}^{k} \alpha!^{\sigma} m_j!^{2s} \\
&= C_\theta A \rho^{\alpha+2m} |t|^{-\alpha-2m} \alpha!^{\sigma} m!^{2s} \sum_{k=0}^{\infty} \frac{(C_\theta |a| A)^k}{k!} \\
&\qquad \times \sum_{\substack{m_0+\cdots+m_k=m \\ \alpha_0+\cdots+\alpha_k=\alpha}} \left\{ \frac{m!}{m_0!\cdots m_k!} \right\}^{1-2s} \left\{ \frac{\alpha!}{\alpha_0!\cdots\alpha_k!} \right\}^{1-\sigma} \\
&\leqslant C_\theta A (2\rho)^{\alpha+2m} |t|^{-\alpha-2m} \alpha!^{\sigma} m!^{2s} \sum_{k=0}^{\infty} \frac{(C_\theta |a| A)^k}{k!} \sum_{\substack{m_0+\cdots+m_k=m \\ \alpha_0+\cdots+\alpha_k=\alpha}} 2^{-\alpha-m} \\
&\leqslant C_\theta A (2\rho)^{\alpha+2m} |t|^{-\alpha-2m} \alpha!^{\sigma} m!^{2s} \sum_{k=0}^{\infty} \frac{(C_\theta |a| A)^k}{k!} \left\{ \sum_{p=0}^{\infty} 2^{-p} \right\}^{2k+2} \\
&\leqslant 4C_\theta A (2\rho)^{\alpha+2m} |t|^{-\alpha-2m} \alpha!^{\sigma} m!^{2s} \sum_{k=0}^{\infty} \frac{(4C_\theta |a| A)^k}{k!} \\
&= \Big(4C_\theta A \exp(4C_\theta |a| A) \Big) (2\rho)^{\alpha+2m} |t|^{-\alpha 2-m} \alpha!^{\sigma} m!^{2s},
\end{aligned}
$$

which is desired. This completes the proof. □

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